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ABSTRACT

This collection of lessons seeks to teach mathematical concepts associated with rates of change. Lessons are presented in the context of biomedical situations. Each section contains several readings relating to the central concept of the section. Example problems are presented in each section and solutions are presented and explained. Each section contains a set of problems for solution by the student. (RE)

BIOMEDICAL MATHEMATICS

UNIT VI

RATES OF CHANGE

STUDENT TEXT

REVISED VERSION, 197/

THE BIOMEDICAL INTERDISCIPLINARY CURRICULUM PROJECT

SUPPORTED BY THE NATIONAL SCIENCE FOUNDATION

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SECTION 1: THE MATHEMATICAL SYMBOLS USED IN THE STUDY OF RATES

1-1 Rates of Change

The world is constantly changing. Few things are as certain as change. The historical increase in the rate of cultural change in the world has been related to the evolution of technology. The number of years between major technological innovations has decreased. New products and ideas become obsolete more quickly than before. It seems clear that technology is capable of a faster rate of change than human institutions can change to accommodate them. School is a human institution, therefore it is inevitable that it is behind the times a little. You no doubt realize that some of what you learn in school will not be relevant to the world you will live in as an independent adult.

This doesn't imply that going to school is a waste of time. There is no way to tell for sure which things will be useful and which things won't. The more you learn, the more useful knowledge you will pick up. The more useless knowledge also. More important - you can learn how to learn - the definitely human adaptive mechanism.

Anyway, from an applied mathematical point of view, few topics are more justifiable than rates of change. Good mathematical descriptions of rates will allow us to project today's trends into the future. For example, we can project world population growth and ponder the implications. We will be able to calculate what will happen to the buying power of money at given rates of inflation. From a biomedical point of view, good health habits can be thought of as slowing down the rate of aging of the body. In automobile accidents the rate of change of the body's momentum is a major factor in whether an injury will occur. In lung function testing the maximum rate that air may be forced from the lungs is more important than the maximum amount of air that may be expired. The list could go on and on.

1-2 Identification of the Changing Variables

In this topic, just as in other mathematical topics, there is a certain amount of basic groundwork to be laid. We are talking about becoming familiar with some mathematical symbols and language.

To illustrate what we mean, consider the case of an automobile cruising down a straight highway at a constant speed. Many things are changing. Can you identify some of them? For one, time is certainly not standing still. Time is relentlessly moving forward. Time will generally be a factor in the study of rates of change. Mathematicians and scientists have agreed to use the symbol "Δt" (pronounced, "delta tee") to represent time intervals. In addition to time, many other things are changing, for instance the position of the car, the total number of revolutions made by the engine, the orientation of the wheels, and so forth.

This is typically the case. That is, for a given situation many things are changing at once. However, to make our analysis simpler we will always select just one changing quantity at a time.

Now we'll go back to our car example. Let's focus our attention on the change in position of the automobile. The change in the position of the car may be measured in units of length. Change in position with time is called speed. As you already know, speed may have many different units--for example, miles per hour, meters per second or perhaps even furlongs per fortnight. There are many other kinds of rates besides speed--for example, miles per gallon, dollars per kilowatt-hour, dollars per kilogram, blood-alcohol concentration decrease per hour.

For the purposes of our study, we are going to concentrate on time-dependent rates--things that change with time. Such rates make up the most important category of rates of change.

We will need some ground rules. We are going to call time intervals Δt. In a time-dependent rate we will always be measuring how fast something is changing with time. For the present we will call the change in the other quantity Δy (pronounced "delta y").

PROBLEM SET 1:

1. Identify possible Δy's and Δt's in this list of units.

    a. seconds
    b. kilometers
    c. hogsheads
    d. weeks
    e. degrees Kelvin
    f. kilograms
    g. centuries
    h. volts
    i. liters
    j. hands
    k. millemia

State a measurable quantity besides time that will change for each of the situations described in Problems 2 through 25. Many have more than one correct answer.

2. A light switch is flipped.
3. A radio is turned down.
4. The accelerator of a car is depressed while the car is on a flat road.
5. The fire under a pot of water is turned off.
6. A cracker is chewed.
7. Elmo begins to fast.
8. An infant grows.
9. Lou Kosite dies.
10. Lem Fosite gets excited.
11. Haley Tosis wakes up in the morning.
12. Hot water is poured into a tub of cold water.
13. A machine replaces some employees in a factory.
14. Taxes go up.
15. The oil is changed in a car.

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16. Fat Ann Alysis goes on a diet.
17. Kar D. Ack runs a kilometer.
18. Felton Farquar eats dinner.
19. A mountain range grows.
20. The continents drift.
21. An ice age occurs.
22. More people are X-rayed.
23. People all over the world use DDT.
24. The world burns oil.
25. People are born.

SECTION 2: AVERAGE RATES

2-1 Elmo's Trip

In the previous section we discussed the idea that a rate of change always involved two variables, one of which was generally time, or Δt. The other quantity we called Δy. The quotient $\frac{\Delta y}{\Delta t}$ measures rates of change. For example, suppose Elmo traveled 40 miles in his automobile and that it took him one hour.

$$\Delta y = 40 \text{ miles}$$

$$\Delta t = 1 \text{ hour}$$

$$\frac{\Delta y}{\Delta t} = \frac{40 \text{ miles}}{1 \text{ hour}}$$

$$= 40 \frac{\text{miles}}{\text{hour}} = \text{rate of change of position}$$

$$\approx 67 \frac{\text{km}}{\text{hr}}$$

You should be familiar with the units of this rate. They are miles per hour, or mph. It is what speed limits are stated in. It is what speedometers in the U.S. are calibrated in. In the rest of the world speedometers are scaled in terms of kilometers per hour. So are speed limits. We are scheduled to catch up with the rest of the world in 1978 when we convert to metric measurements.

What is the significance of the 40 $\frac{\text{miles}}{\text{hour}}$ result of our simple-minded example? It is the average rate for the trip. It says nothing about how fast Elmo was going at any particular instant in the one-hour interval. Suppose the 40 miles represented a typical commuting trip. Then some time was probably spent stopped at a stoplight or grocery store. The rate at this time was 0 $\frac{\text{miles}}{\text{hour}}$. Probably some time was spent at a freeway speed of 55 mph. The result of all this stopping and going was that Elmo covered 40 miles in one hour. Therefore, his average rate for the entire journey was 40 miles per hour. This definitely does not imply that he spent one hour traveling at an absolutely constant rate of 40 mph.

This is a good place to sound a cautionary note. Many students get the erroneous idea that average rates are calculated like averages. They are not. We did not get Elmo's average rate by adding (55 + 0) and dividing by 2. It is unfortunate that the word "average" is used to describe two concepts which are not calculated in the same manner.

You probably expected something funny in this section because Elmo is here. Let this teach you that nothing is certain. Especially in a unit on change.

2-2 Deltas are Differences

Seldom in a real-life situation is a rate calculation as straightforward as in the previous section. The distance traveled generally comes from a pair of odometer readings. The odometer registers the total number of miles on a car. For example, suppose the odometer read 94,363.8 at the start of the trip and 94,403.8 at the end of the trip. What would Δy be? Δy is the difference between the two readings.

$$\Delta y = 94{,}403.8 - 94{,}363.8$$

$$\Delta y = 40 \text{ miles}$$

Notice that Δy is the difference of two numbers. Remember that this is the reason we use delta or Δ. It is easily associated with differences. The words "delta" and "difference" both start with the letter "d" which is supposed to remind you that Δy and Δt are both actually differences. The difference associated with Δt in our example is the difference between the starting time and the time when Elmo stopped. For example, if Elmo started his trip at 6:31 a.m. and finished it at 7:31 a.m., then Δt is the difference between these two times, or 1 hour.

2-3 A More Complicated Example

PROBLEM:

Y. Noe lives on skid row. One day, by 1:00 P.M., he had bummed enough money for a big bottle of burgundy. By 1:30 P.M. he had finished off the bottle. At 2:00 P.M. his blood-alcohol concentration had reached its maximum value of .24 per cent. By 2:00 A.M. the next morning his blood-alcohol concentration was .08 per cent. What was the average rate at which Mr. Noe's blood-alcohol concentration decreased?

SOLUTION:

In this problem Δy is going to be the change in blood-alcohol concentration stated in terms of per cent alcohol.

$$\Delta y = .24\% - .08\%$$

$$\Delta y = .16\%$$

Δt, as usual, is the time interval.

$$\Delta t = 2{:}00 \text{ A.M.} - 2{:}00 \text{ P.M.}$$

$$\Delta t = 12 \text{ hr}$$

The average rate of decrease is $\frac{\Delta y}{\Delta t}$.

$$\frac{\Delta y}{\Delta t} = \frac{.16\%}{12 \text{ hr}}$$

$$= .0133\ldots\frac{\%}{\text{hr}}$$

PROBLEM SET 2:

1. Dolores Mhoney thought that she was spending too much money, so she started to keep track of her monthly expenditures. By day's end on December 3 she had spent

$38. By day's end on December 15 she had spend $158. What was Dolores' average rate of expenditure in terms of dollars per day during this period?

2. Doctor Dave told Nurse Naomi to meter the IV (intravenous) solution of 5% dextrose into Patient Paul at the rate of .5 liter per hour until it was all gone. When she hung up the bottle it read .95 liter. When she came back ten minutes later, it read .85 liter.

a. How fast was the solution entering Patient Paul's body? State the result in units of liters per hour.

b. Should Nurse Naomi slow down or speed up the rate at which the solution enters Patient Paul's body?

3. Perry Cardium had a serious case of bacterial endocarditis; therefore, Doctor Denise prescribed a massive dose of penicillin, to be given intravenously over a period of 24 hours. The drug was dissolved in one liter of solution. This liter of solution was to be metered into Perry over a 24 hour period. When the bottle was first hung, it read "1000 ml." One hour later it read "940 ml."

a. What was the rate at which the solution was entering Perry's body? State the answer in units of ml per day.

b. Should the rate be increased or decreased?

4. Fernly's electric meter read 58,942.3 kilowatt-hours on May 3 and 58,972.7 kw-hr on May 19. What was Fernly's average rate of electricity usage (in kw-hr per day) during that period?

5. During the period Jan. 1, 1920, to Jan. 1, 1970, the population of the United States rose from 105,710,620 to 203,211,926. What was the average rate of population increase (in people per year) during that period?

6. On June 1 it took $10.00 to buy a market basket of food. Four months later it took $10.30 to buy the same amount of food. What was the rate of inflation for the original market basket? State your answer in terms of cents per year.

7. Mrs. Klackplug could not convince her husband that he was an outrageous snorer. Therefore she decided to stay up one night and record the number of snores. The period between 1:31 a.m. and 2:06 a.m. was a period of very high snoring incidence. During this period Mrs. Klackplug's tally rose from 132 snores to 307 snores. What was Mr. Klackplug's average snoring rate (in snores per minute) during this period?

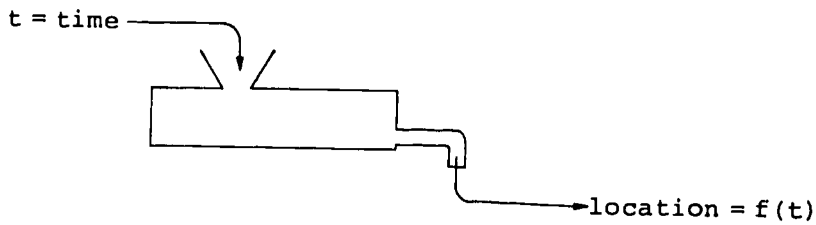
8. M. Phee Sema performed a pulmonary function test. When Doctor Donna analyzed the laboratory report on this test she observed that the volume of expired air increased from .5 liters to 2.0 liters between t = .5 seconds and t = 1.7 seconds. What was Ms. Sema's average rate of air expiration in liters per second during this period?

9. Norman Mall performed a pulmonary function test. When Nurse Norbert examined the laboratory results, he noted that the volume of expired air increased from 1.1 liter to 2.6 liters from t = .6 seconds to t = .9 seconds. What was Norman's average rate of air expiration during this period?

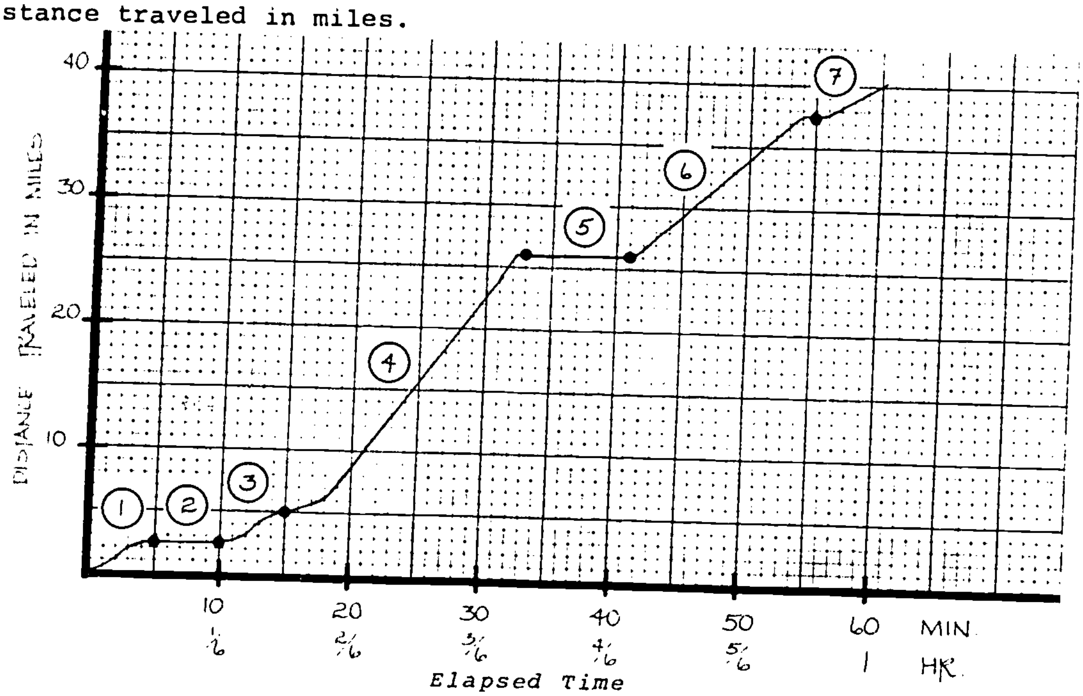
SECTION 3: RATES--GRAPHS AND SLOPE

3-1 Rates and Graphs

Suppose that for some reason we were still interested in Elmo's trip. It is possible to analyze his trip in a more detailed manner. We can view the automobile trip as setting up a functional relationship between time and distance. All this means is that if more accurate records of the trip were available, then knowledge of time would be sufficient to determine Elmo's location (i.e., distance from starting point) at any particular moment. We can represent this idea by a function machine.



As you know by now, whenever we start talking about functions, we generally try to graph them. Below is a graph that represents Elmo's trip. On the horizontal axis we have elapsed time in both minutes and hours. On the vertical axis we have distance traveled in miles.



This is a log of Elmo's activities during each leg of the journey.

1. Elmo drives from work to the grocery store.
2. Elmo picks up a few things at the store.
3. Elmo drives to the freeway.
4. Elmo drives on the freeway.
5. Elmo is stopped for speeding.
6. Elmo continues on, somewhat chastened.
7. Elmo drives from the freeway to his home.

The first thing to notice is that Elmo's average speed for the entire trip is still 40 mph. He traveled 40 miles in one hour. For the entire trip $\Delta y = 40$ miles and $\Delta t = 1$ hour. In a similar fashion, we can calculate the average speed for each leg of the trip. Let's calculate Elmo's average speed just before he was stopped for speeding. Leg number 4 (between points A and B) occurred between t = 33 min and t = 15 min. Therefore, $\Delta t = (33 - 15)$ or 18 min. In this 18-minute interval Elmo's distance traveled increased from 5 miles to 26 miles. Therefore $\Delta y = (26 - 5)$ or 21 miles.

$$\frac{\Delta y}{\Delta t} = \frac{21}{18}\,\frac{\text{miles}}{\text{minutes}}$$
$$= \frac{7}{6}\,\frac{\text{miles}}{\text{minutes}}$$

We can convert miles per minute to miles per hour by multiplying by the conversion factor, $60\ \frac{\text{min}}{\text{hr}}$.

$$\frac{\Delta y}{\Delta t} = \frac{7}{6}\,\frac{\text{miles}}{\text{min}} \cdot 60\ \frac{\text{min}}{\text{hr}}$$
$$= 70 \text{ mph}$$
$$\approx 98 \text{ kph}$$

It's clear why Elmo was stopped for speeding.

The following are the average speeds for each of the legs of Elmo's trip.

1. 30 mph (42 kph)
2. 0 mph
3. 30 mph (42 kph)
4. 70 mph (98 kph)
5. 0 mph
6. ~50 mph (70 kph)
7. 30 mph (42 kph)

This is a good place to repeat a point that we made earlier. Average rates are not found by adding up rates and dividing by the number of added rates. For example, the sum of the seven rates listed above is 210. This number divided by 7 is 30. Thirty mph is not the average rate.

3-2 Average Rate and Slope

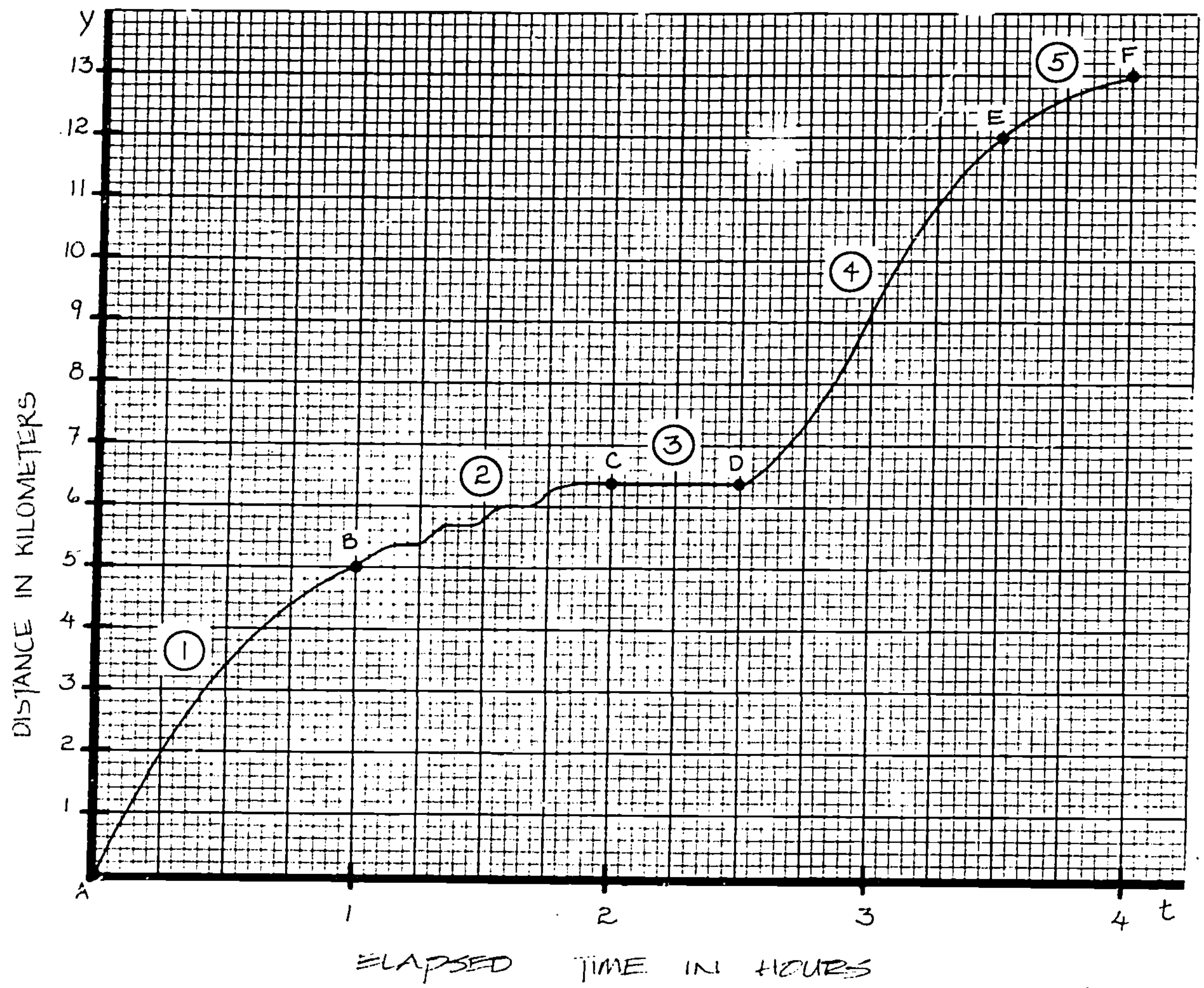
You have probably recognized by now that average rates are closely related to our old friend slope. Elmo's average speed for leg 4 of his trip is identical to the slope of the line between points A and B on the graph. Δy = rise and Δt = run.

$$\frac{\Delta y}{\Delta t} = \frac{\text{rise}}{\text{run}}$$
$$= \text{slope}$$

Steeper lines mean greater slopes and faster speeds. On the other hand, a horizontal line has a slope of zero which is equivalent to saying that Elmo was traveling at a speed of 0 mph.

PROBLEM SET 3:

Gaylord Sky likes to hike. Below is a graphical record of his hike. Problems 1 through 4 refer to this graph.



The following is a description of each leg of Gaylord's hike.

Leg #

1 (A to B) Gaylord starts hiking, full of energy. He gradually slows down as the slope increases.

2 (B to C) Gaylord hikes up a very steep slope to a pass. Each short period of walking is followed by a short rest.

3 (C to D) Gaylord stops to eat lunch.

4 (D to E) Gaylord rapidly descends the other side of the pass.

5 (E to F) Gaylord walks slowly along the floor of a valley to a lakeside campsite.

PROBLEMS:

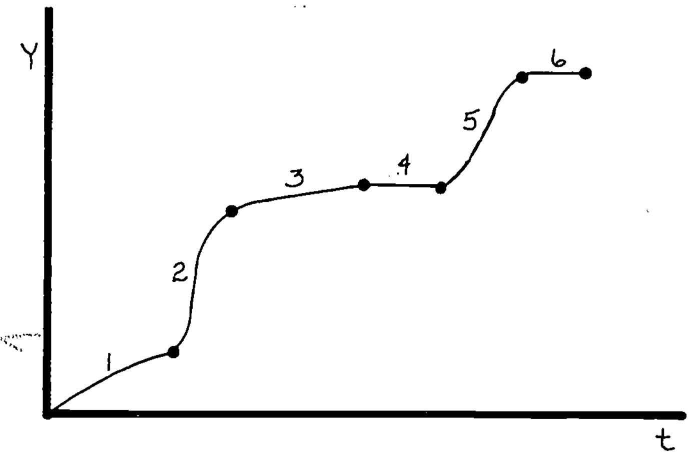
1. Calculate the average rate in $\frac{\text{km}}{\text{hr}}$ for each leg of the hike. (5 answers)
2. Calculate the average rate for the entire trip.
3. Which leg has the greatest slope?

4. Which leg has the smallest slope?

5. In general, suppose leg Y has a greater slope than leg Z; therefore the average rate for leg Y is (greater, less) than leg Z.

6. Identify the leg with the greatest average rate.

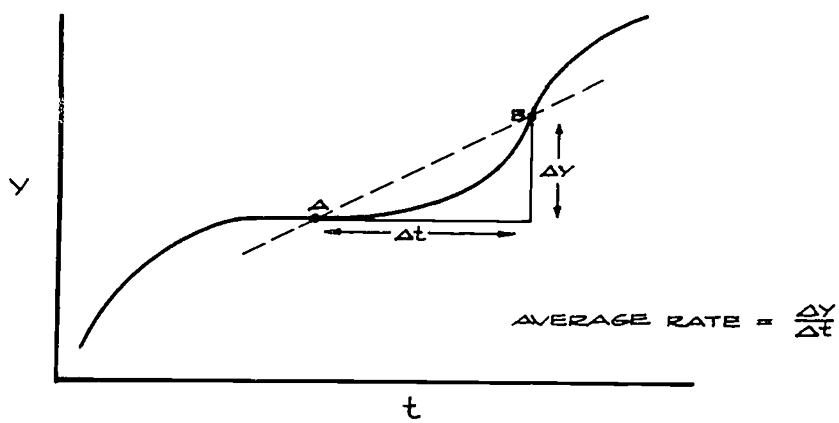
7. Identify legs with an average rate of zero.



SECTION 4: THE RELATIONSHIP BETWEEN A DESCRIPTIVE LINEAR EQUATION AND RATE OF CHANGE

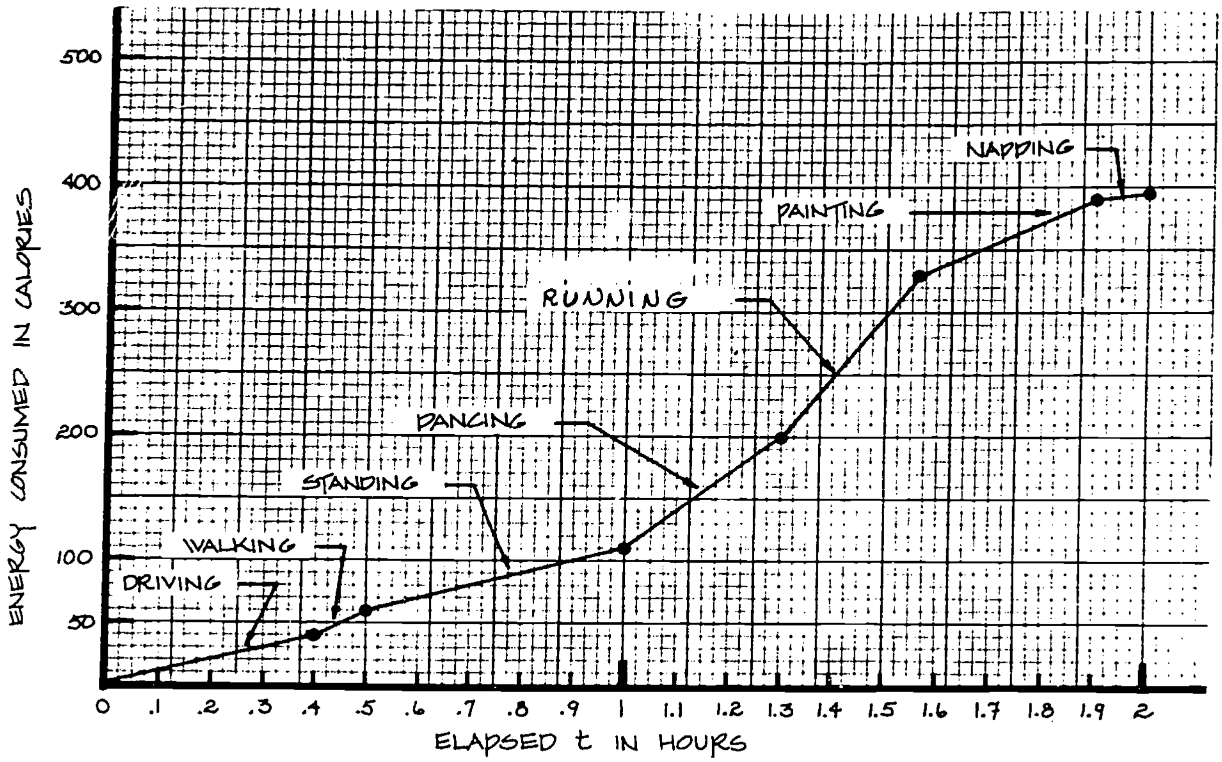
4-1 Elmo Goes to a Party

In the previous section Elmo drove home from work. We represented distance traveled as a function of time. We found the average rate of change of distance or, in other words, the average speed by computing $\frac{\Delta y}{\Delta t}$. Δy and Δt are shown in the diagram below.



We found that the average rate between points A and B was identical to the slope of the line between points A and B. All of this is very important to master. Therefore, this section will deal mainly with reinforcing these ideas in relation to new examples. The graph on the following page shows Elmo's energy consumption during the course of a party. We constructed the graph by referring to the table on page 84 of Unit II Biomedical Science Laboratory Manual and assuming that Elmo has a mass of 60 kg (about 132 lb).

Elmo drove to the party, walked to the house and then stood around. After a while a charitable female coaxed him to dance. Then somebody bet him ten dollars that he wouldn't run around the block. Elmo took him up on the bet. When he returned, the guy was gone. Whereupon, Elmo decided to do something more sedate. He played around with some Day Glo paints that were available and then took a nap on the couch.



We can easily calculate the average energy expenditure for the entire evening. From the time Elmo set out for the party until the end of his nap, 2 hours passed. During that period 395 Calories were expended. Therefore Δy = 395 Cal and Δt = 2 hours.

$$\text{Average rate of energy expenditure} = \frac{395}{2}\ \frac{\text{Cal}}{\text{hr}}$$

$$= 197.5\ \frac{\text{Cal}}{\text{hr}}$$

4-2 Dancing and Running

Similarly, we can calculate the average rate of energy expenditure for any sub-interval we choose. Let's calculate the average rate while Elmo was dancing and running. From the graph we see that Elmo started dancing at t = 1.0 hour. He finished running at t = 1.55 hour. Therefore, Δt = .55 hour. During this time interval Elmo's energy expenditure increased from 110 to 330 Cal. Therefore, Δy = 330 - 110 or 220 Cal.

$$\frac{\Delta y}{\Delta t} = \frac{220}{.55}\ \frac{\text{Cal}}{\text{hr}}$$

$$= 400\ \frac{\text{Cal}}{\text{hr}}$$

4-3 Explicit Equations Relating y and t

The following seven equations describe Elmo's seven different activities at the party.

	ACTIVITY	EQUATION	DOMAIN
1.	Driving	$y = 100t$	$0 \le t \le .4$
2.	Walking	$y = 200t - 40$	$.4 \le t \le .5$
3.	Standing	$y = 100t + 10$	$.5 \le t \le 1$
4.	Dancing	$y = 300t - 190$	$1 \le t \le 1.3$
5.	Running	$y = 520t - 476$	$1.3 \le t \le 1.55$
6.	Painting	$y \approx 171t - 64$	$1.55 \le t \le 1.9$
7.	Napping	$y = 50t + 295$	$1.9 \le t \le 2.0$

Notice that each of the equations is in the form $y = mt + b$. The slope, or m, is the average rate over the given domain. For example, one equation which describes Elmo's energy expenditure is

$$y = 100t$$

For this equation $m = 100$; therefore the average rate of energy expenditure was 100 Cal per hr.

PROBLEM:

What was the average rate of energy expenditure while Elmo was running around the block?

SOLUTION:

The equation which describes this portion of the graph is

$$y = 520t - 476$$

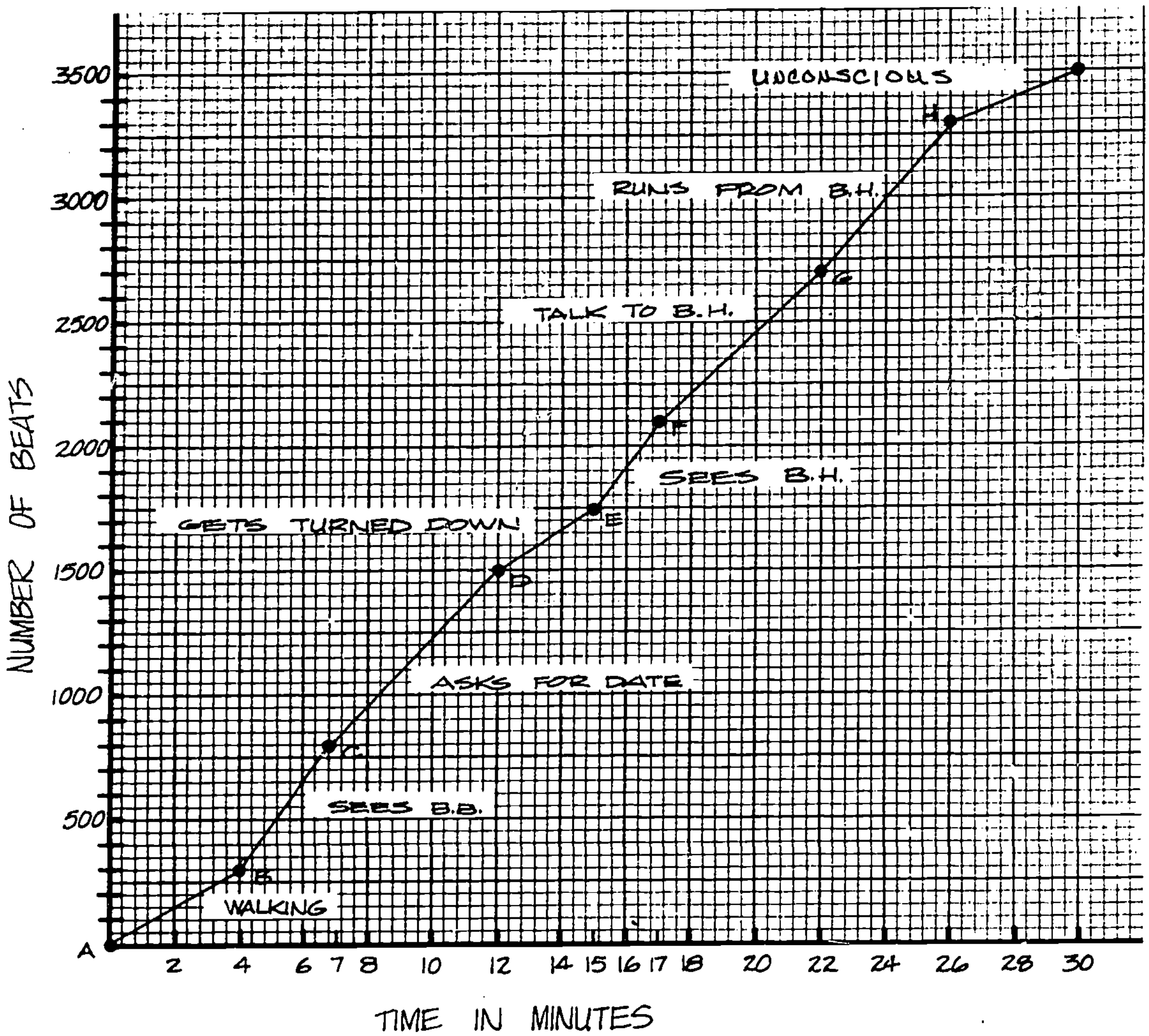
The slope of this equation is 520; therefore the average rate was 520 Cal per hr.

Below we have listed the average rate for each of Elmo's activities. Notice that the average rate is simply m in the corresponding equation.

	ACTIVITY	AVERAGE RATE (Cal/hr)
1.	Driving	100
2.	Walking	200
3.	Standing	100
4.	Dancing	300
5.	Running	520
6.	Painting	171
7.	Napping	50

PROBLEM SET 4:

When Elmo was a high school student he walked to school. Below we have a record of a particularly disastrous stroll for our hero. On the vertical axis of the graph on the next page, we have recorded Elmo's total number of heartbeats. On the horizontal axis we have the elapsed time in hours. We can use the information on the graph to calculate Elmo's average pulse rate during various events.



The Story

Elmo was in love with Bonita Bash. Bonita's house was on his way to school. Elmo had been screwing up his courage to ask her for a date. He timed his departure so that he would meet Bonita coming out of her house on her way to school. When he got close to Bonita Bash's house his heart began to pound. Would she come out as planned? She did! His timing was almost perfect. He greeted her and made nervous small talk while working himself up to the main question.

"Uh,...m can you go to the football game with me on Friday?" he asked.

"No, I already have a date," she said.

"Well, how about a movie Saturday?" he asked. No, she felt a cold coming on and would probably not be feeling very well on Saturday. Elmo still hadn't quite gotten the message.

"O.K., then, how about going bike riding on Sunday afternoon?" No, she considered Sunday to be a day of rest. She preferred to spend it quietly at home with her family.

Elmo shut up. He might not know much about females, but he did know a little about baseball. In baseball, three strikes means that you're out. They walked quietly together for a short distance. They turned a corner and there was Bull Heavy. It was widely known that Bull was trying to date Bonita. Bull was unhappy to see Elmo with Bonita. Elmo was unhappy to see Bull. Bull and Elmo had a little discussion. It turned out that Bull thought that Elmo and he should settle their differences in a physical way. Elmo didn't see things that way. Bull took a swing at Elmo, and Elmo responded as bravely as he could. He ran toward school. Unfortunately, he tripped and fell, knocking himself out cold in the process.

1. What was Elmo's average pulse rate for the entire episode (in beats per minute)?

2. What happened to Elmo's pulse rate after he struck out with Bonita? Justify your answer by referring to the graph. No calculation is necessary.

3. What happened to Elmo's pulse rate when he turned and ran away from Bull Heavy?

4. When was Elmo's pulse rate the slowest?

5. What was the average pulse rate during the period from when he first saw Bonita until he struck out? (B to D on the graph)

6. What was Elmo's average pulse rate while he was running? (G to H)

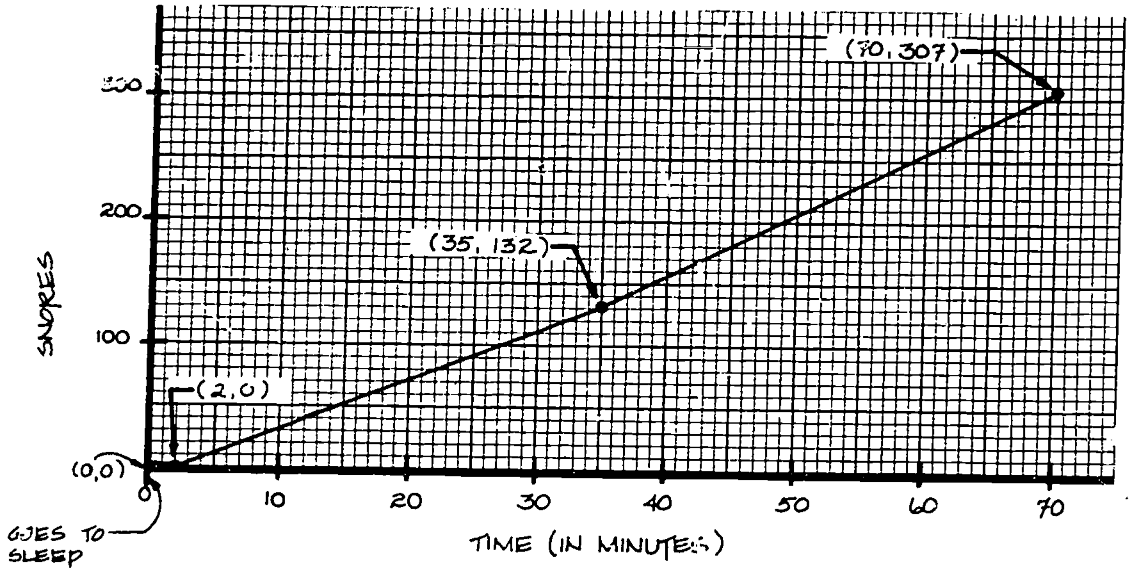
7. The eight equations below describe different portions of Elmo's walk. Order the equations from slowest heartbeat to fastest.

$$y = 75t$$
$$y = 167t - 367$$
$$y = 140t - 180$$
$$y = 83t + 500$$
$$y = 175t - 875$$
$$y = 120t + 60$$
$$y = 150t - 300$$
$$y = 50t + 2000$$

8. The equation $y = 75t$ describes the initial portion of the graph when Elmo was walking. What are the domain and range of validity? In other words, for what values of y and t does the equation describe the graph?

9. On the following page we have Mrs. Klackplug's graph of the number of snores her husband emitted as a function of time.

   a. How many minutes passed before Mr. Klackplug snored his first snore?
   b. Write three linear equations that describe the graph.
   c. State the domain of validity for each equation.



SECTION 5: INSTANTANEOUS RATES OF CHANGE

5-1 Elmo Falls Out of an Airplane

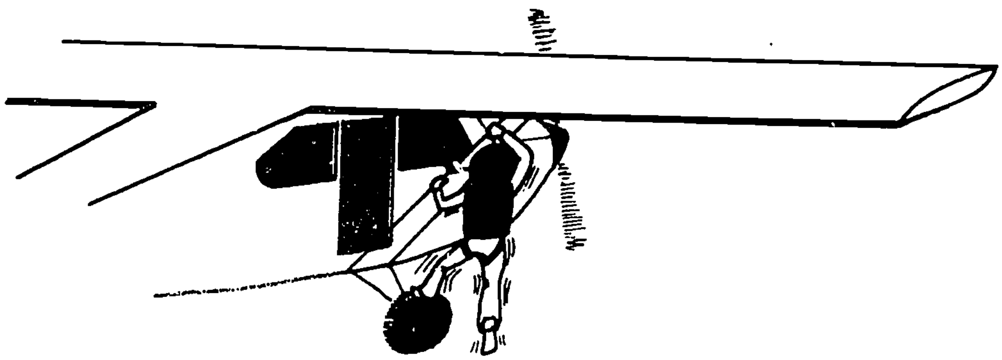
After Elmo's humiliating experience with Bull Heavy, he decided that he needed to do something brave to give his self-image a tune-up. After thinking about it for a couple of weeks, he decided that sport parachuting was the thing for him. So one weekend he went to Flay Kee's Sky Diving School and laid down $40 that he had saved from his paper route for instruction and his first jump. He spent the better part of a day learning the basics of parachuting. By late afternoon his instructor had declared that he was "ready to go." Elmo was a little nervous. He did not appreciate the instructor's choice of words.

Typically a student's first jump is on a "static line." That is, the rip cord of his main parachute is attached to a long cord which is attached to the airplane. Consequently, the rid cord is pulled automatically when the parachutist leaves the airplane. Elmo was a little hesitant in boarding the aircraft; therefore, he was the last one in. He didn't realize that this would mean that he would have to be the first one out.

As the aircraft took off and approached drop altitude, Elmo became more and more terrified. He began to think of ways to chicken out. But in vain. There was no honorable way out. No dishonorable way either. There was no way that the others could jump if he didn't go out the door first. At some point he resigned himself to his fate. He was sure that he was going to die. He felt sick. He had finally convinced himself that he was stupid, crazy, imbecilic, etc.

He felt a tap on his shoulder that meant that he should go out the door, hang onto the wing strut, and put exactly one foot on the wheel.

In a dazed state, Elmo did what he had been trained to do while on the ground. When he was in position, he had a moment for reflection. He noticed that the wheel, where his foot was, was free to rotate. This struck Elmo as being very unstable. Why, a person could fall, thought Elmo. Then it occurred to him that this was the reason he was 2000 feet (about 600 m) above the earth, hanging onto a wing strut. Somehow this thought had a calming effect.

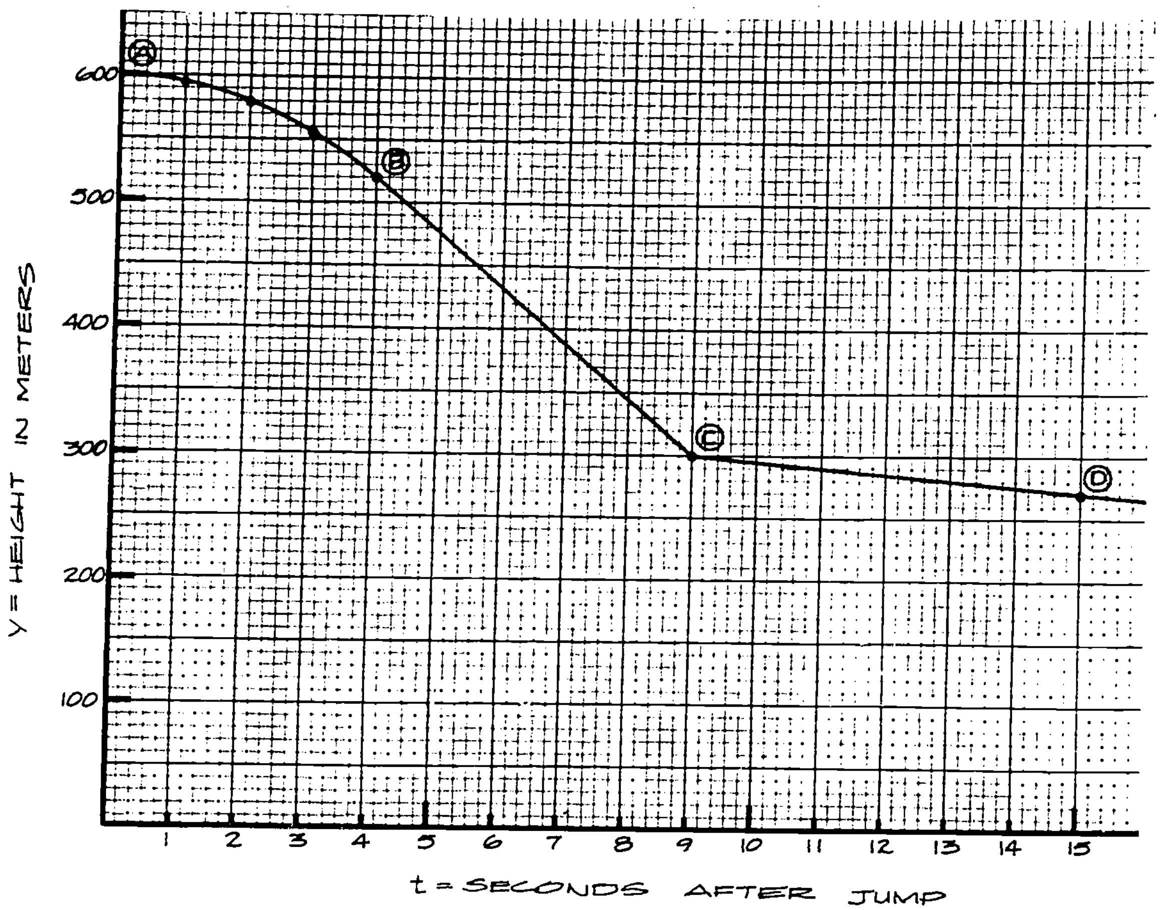


He felt a tap on his left hand. Elmo looked up from the ground and over to the jumpmaster. He was violently stabbing his thumb at the ground. "That could mean only one thing," muttered Elmo. "It's time to die...er, dive." Elmo let go of the airplane and began counting. "One thousand one, one thousand two...one thousand six...ONE THOUSAND SIX?!!" He remembered that the 'chute was supposed to open no later than 1004. He looked up. There was his static line fluttering in the breeze. Flay Kee had neglected to attach it to the plane. Elmo would have liked to think about what to do for a couple of hours. But somewhere in the back of his mind he realized that haste was important. He pulled the ripcord of his emergency chute and floated the balance of the way to the ground uneventfully.

5-2 Elmo's Altitude as a Function of Time

The graph, on the following page, shows Elmo's altitude as a function of elapsed time in seconds after letting go of the airplane. After Elmo left the plane he gradually picked up speed. The curved portion of the graph from A to B describes this portion of his fall. After a certain period of time his velocity no longer increased, the force of gravity was counteracted by the friction of the air. Elmo's speed was constant during this portion of his fall. The line from B to C describes this phase, called the "terminal velocity" phase. After 9 seconds, his parachute opened and his rate of descent was drastically reduced.

By now, you should be able to calculate the average rate of fall for any given time period. We now wish to extend the concept of rate to include the idea of an instantaneous rate. It is one thing to ask, "What was Elmo's average rate of fall between 5 and 9 seconds after leaving the airplane?" It is quite another thing to ask, "How fast was Elmo falling at $t = 7$ seconds?" This question asks for an instantaneous rate, not an average rate. In order to answer the second question we will have to describe the graph analytically, that is, in terms of an equation.



Notice that the graph appears to be linear between t = 4 seconds and t = 9 seconds. We will write an equation of the form y = mt + b for this linear function of time.

By following our standard procedures we get the equation

$$y = -44t + 696$$

for the linear portion of the graph between B and C. Notice that the slope is negative 44. We interpret this to mean that Elmo's altitude was decreasing at the rate of 44 meters per second. This is quite fast. It's roughly equivalent to falling the length of a football field in 2 seconds. The domain of validity for this equation is $4 \leq t \leq 9$ seconds.

5-3 Instantaneous Rates

Previously, when we calculated average rates, we divided Δy by a non-zero Δt. The determination of an instantaneous rate is more subtle. The following argument should convince you of this fact.

How long is an instant? By definition, an instant is no time at all, or 0 seconds. Therefore, $\Delta t = 0$ seconds, and we are in trouble already. We cannot calculate the instantaneous rate in a similar fashion to average rate, because that would imply division by zero, a no-no.

$$\text{Average rate} = \frac{\Delta y}{\Delta t}$$

$$\text{Instantaneous rate} = \frac{\Delta y}{0} \text{ ?? } \underline{\text{NO}}$$

How much can anything change in a time interval of zero seconds? Clearly nothing can change at all in no time at all; therefore, $\Delta y = 0$. This leads to the following mysterious expression.

$$\text{Instantaneous rate} = \frac{\Delta y}{\Delta t}$$
$$= \frac{0}{0} \text{ !?}$$

If all of this is confusing, don't feel bad. It confused the ancients also. Up until the time of Isaac Newton, the problem of how to deal with instantaneous rates was in a state of confusion. Newton's approach was first to try to write an explicit equation for the average rate for a given situation. Then he observed what happened to the average rate as the time interval, or Δt, got smaller and smaller. For linear functions this is straightforward. For example, we know that

$$\frac{\Delta y}{\Delta t} = m$$

We can see from this formula that any choice of Δt will not alter m. Therefore, when $\Delta t = 0$ and $\Delta y = 0$, $\frac{\Delta y}{\Delta t}$ is the constant m. Therefore, for linear functions, the instantaneous rate and average rate are identical.

Now we return to the specific case of Elmo falling at "terminal velocity," the maximum velocity of an object falling through air. We found that his height (y) at any time (t) was given by the equation

$$y = -44t + 696$$

in the time interval $4 \leq t \leq 9$ seconds after jumping out of the airplane.

PROBLEM:

How fast was Elmo falling at t = 7 seconds?

SOLUTION:

We reason as follows. We see from inspection of the equation that the slope of the line is -44 at any point on the line; therefore, at the instant t = 7 seconds, Elmo was falling at the rate of 44 meters per second.

5-4 $\frac{0}{0}$

The $\frac{0}{0}$ beast deserves some explicit attention. Some mathematicians have given it the name "indeterminate." Others have chosen to call it "useless." This is because the quotient $\frac{0}{0}$ may have any value at all. Recall that we can check an indicated division like

$$\frac{47}{100} = .47$$

by performing multiplication.

Does 47 = (.47)(100)?

Yes, 47 = 47

Since 47 = 47, we assume that the previous statement

$$\frac{47}{100} = .47$$

is also true.

Let's see what happens when we apply a similar argument to the equation

$$\frac{0}{0} = 14$$

$$\text{Does } 0 = 14(0)?$$

$$\text{Yes, } 0 = 0$$

Since 0 = 0, we assume that the first statement is true, or

$$\frac{0}{0} = 14$$

But, we did not have to choose 14. We could have chosen any number at all. For example, suppose

$$\frac{0}{0} = c$$

where c is any constant.

$$\text{Does } 0 = (c)(0)?$$

$$\text{Yes, } 0 = 0$$

And we see that the quotient $\frac{0}{0}$ can be equal to any number at all.

The quotient $\frac{0}{0}$ may take on any value depending on the particular situation that it finds itself in. It is a mathematical chameleon. It is a shifty, slippery character. You might even think of it as the politician of mathematics. It is because of the slippery nature of this quotient that it is impossible to evaluate it directly for any given situation. Following sections will develop indirect methods of evaluating it for a given situation. If it weren't for these indirect methods, it would be impossible to calculate instantaneous rates.

PROBLEM SET 5:

1. The graph on the following page describes the relation between Elmo's altitude and elapsed time while he was under a full parachute canopy. Notice that the scale is different than the graph in the text.

   a. Compute the average rate between points A and D on the graph. Show calculations.

   b. Compute the average rate between points A and C on the graph. Show calculations.

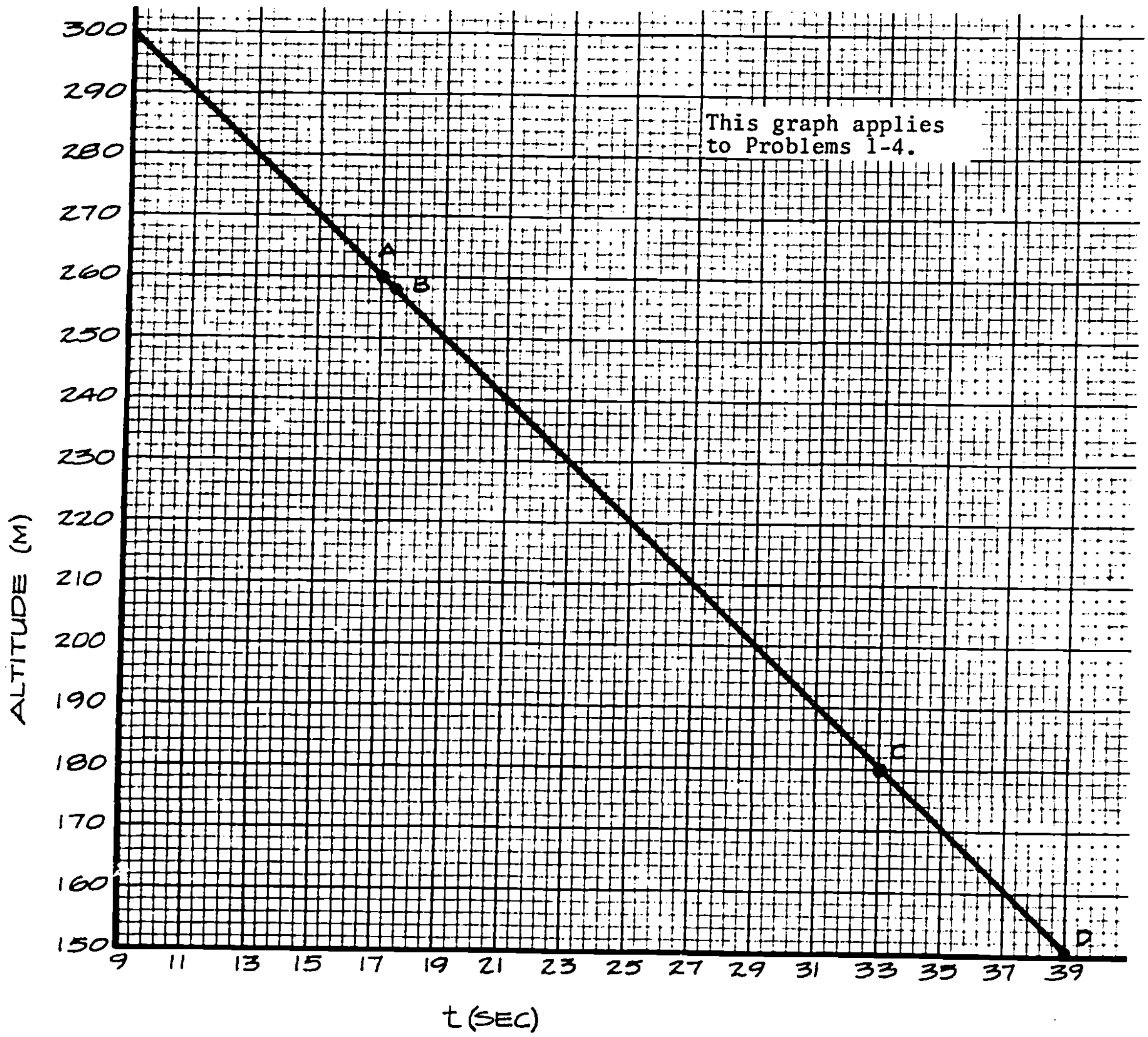
   c. Compute the average rate between points A and B on the graph. Show calculations.

   d. On this linear portion of the graph, the average rate (does, does not) change.

   e. It is (reasonable, unreasonable) to say that Elmo's (average, instantaneous) rate at point A is equal to the average rate between any two points on this graph.

2. What was Elmo's instantaneous rate at t = 14 seconds?

3. What was Elmo's instantaneous rate at t = 20 seconds?



4. Explain why we cannot yet say what Elmo's instantaneous rate was at t = 3 seconds.

Refer to the graph in Section 5-2 to answer Problems 5 - 8.

5. What was Elmo's instantaneous rate at t = 6 seconds?

6. We demonstrated that the equation y = -44 + 696 described the terminal velocity portion of Elmo's fall (B to C on the graph). Had Elmo neglected to open his reserve 'chute, how long would his fall have lasted?

7. a. Derive the equation that describes the open parachute portion of his descent.

b. Calculate the total elapsed time of his fall.

c. Calculate the average rate for the complete trip.

8. Calculate the average rate for the curved portion of the graph in the text.

9. Identify the statements below that can never be true for any situation.

a. $\frac{0}{0} = -1$

b. $\frac{0}{0} = 20$

c. $\frac{0}{0} = 5,376.4$

d. $\frac{0}{0} = \pi$

e. $\frac{0}{0} = -9,746,387$

f. $\frac{0}{0} = \sqrt{2}$

g. $\frac{0}{0} = .333...$

h. $\frac{0}{0} = .0000002$

10. The following are the equations that describe Elmo's heartbeat during the Bonita Bash episode.

EQUATION	DOMAIN OF VALIDITY
$y = 75t$	$0 \le t \le 4$
$y = 167t - 367$	$4 \le t \le 7$
$y = 140t - 180$	$7 \le t \le 12$
$y = 83t + 500$	$12 \le t \le 15$
$y = 175t - 875$	$15 \le t \le 17$
$y = 120t + 60$	$17 \le t \le 22$
$y = 150t - 600$	$22 \le t \le 26$
$y = 50t + 2000$	$26 \le t \le 30$

State Elmo's instantaneous heartbeat rate for the following instants.

a. $t = 13$ min

b. $t = 23$ min

c. $t = \pi$ min

d. $t = \sqrt{25}$ min

*e. Describe the problems you encounter when you try to evaluate Elmo's instantaneous rate at $t = 4$ minutes. List all times that have the same problem associated with them.

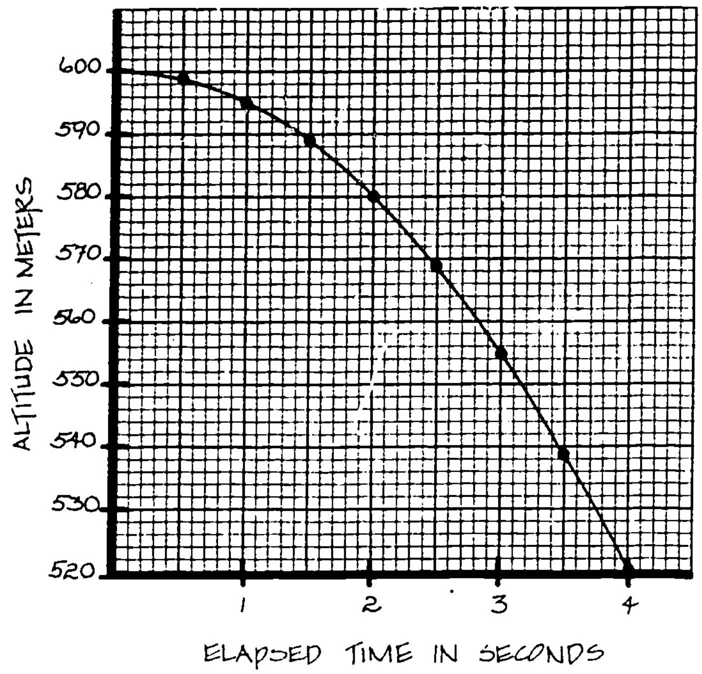
SECTION 6: AVERAGE AND INSTANTANEOUS RATES FOR NONLINEAR FUNCITONS

6-1 Average Rates Near t = 1 Second

Just after Elmo and the airplane parted company, the relationship between Elmo's altitude and elapsed time was not linear. It is on this portion of Elmo's fall that we want to focus attention now. A magnified version of this phase of his fall is on the following page.

You probably recognize this curve. It is a parabola. As you will learn in science, the distance a free-falling body travels is related to elapsed time by the equation

$$s = 5t^2$$



where $\underline{s}$ = distance in meters

$\underline{t}$ = elapsed time in seconds

The t^2 term in the equation is responsible for the parabolic shape of the graph.

Since Elmo started out at 600 m, his altitude at any time is 600 meters minus the distance he has fallen.

$$y = 600 - s$$

Therefore, his altitude at time t is given by the equation

$$y = 600 - 5t^2$$

where y = altitude in meters

t = elapsed time in seconds.

Just as before, we can calculate Elmo's average rate of fall for any given time interval. However, the calculation of his instantaneous rate at a given time will present new problems. An inspection of the graph will reveal that the graph gradually gets steeper with movement to the right. This is equivalent to saying that Elmo's rate of fall gradually increases. Since the slope of the graph is not constant, the average rate and the instantaneous rate will not necessarily be the same. However, we can get better and better approximations to the instantaneous rate by choosing smaller and smaller values for Δt.

To illustrate how we'll do this, we will approximate the instantaneous rate at $t = 1$ sec.

STEP 1: Calculate the average rate between t = 1 sec and t = 2 sec.

In order to obtain Δy we will use the equation

$$y = 600 - 5t^2$$

to calculate y, for two different values of t. The difference of these two values of y will be Δy.

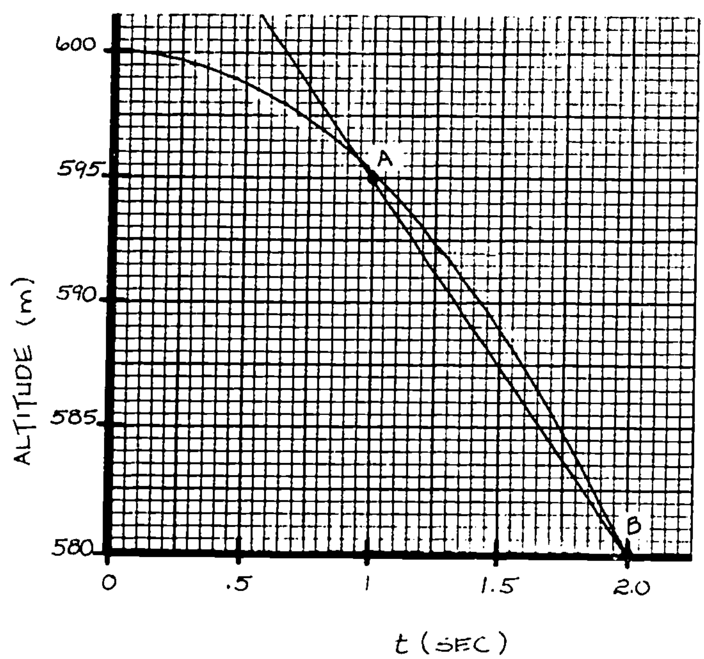
When $t = 1$
$$y = 600 - 5(1^2) = 600 - 5 = 595$$

When $t = 2$
$$y = 600 - 5(2^2) = 600 - 5(4) = 600 - 20 = 580$$

We see that Elmo's altitude dropped 15 m; therefore $\Delta y = -15$.

$$\frac{\Delta y}{\Delta t} = \frac{-15}{1}\ \frac{m}{sec} = -15\ \frac{m}{sec}$$

What we have done here is to calculate the slope of the line between the two points on the graph corresponding to t = 1 sec and t = 2 sec (points A and B below).



STEP 2: Calculate the average rate for the time interval t = 1 and t = 1.2.

When t = 1, y = 595 m, as before.

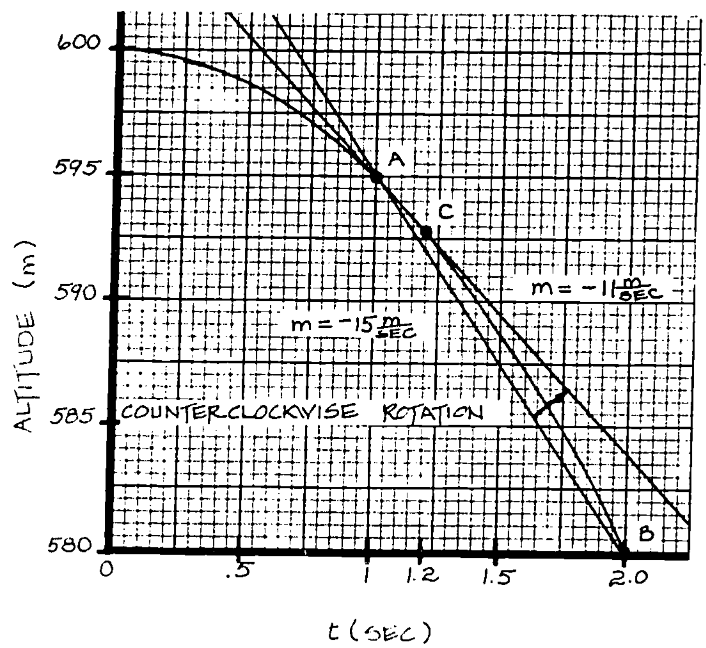
When t = 1.2
$$y = 600 - 5(1.2^2) = 600 - 5(1.44) = 600 - 7.2 = 592.8$$

Elmo's altitude dropped 2.2 m; therefore $\Delta y = -2.2$ m while $\Delta t = .2$ sec.

$$\frac{\Delta y}{\Delta t} = \frac{-2.2}{.2}\ \frac{m}{sec}$$

$$= -11\frac{m}{sec}$$

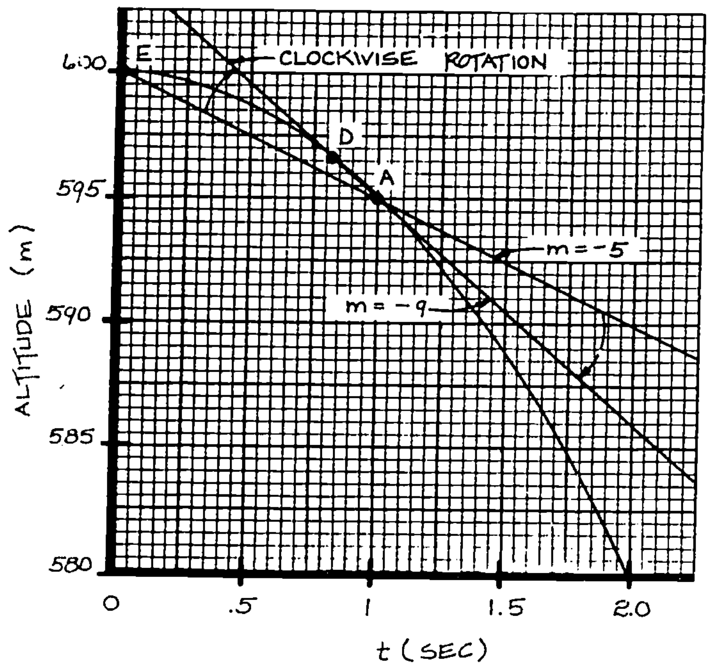
In this step we have calculated the slope between points A and C on the following graph.



It is time to step back and look at what we have done so far. Notice that the line has swung counterclockwise around point A between steps 1 and 2 as indicated in the graph. Now try to imagine what would happen if we moved point C closer to point A. The line would swing around a little more. In other words, as we approach point A from the right the slope increases (i.e. becomes less negative). We can make a similar approach to the instantaneous rate from the other side of 1 sec. In the next illustration we have graphed two lines similar to the ones related to steps 1 and 2. (See graph on the following page.)

Notice that the line that goes through A and D is rotated clockwise from the line that goes through A and E. What do you think would happen to the slope if point D was moved a little closer to A? The line would rotate a little more clockwise. In other words, the slope would decrease.

All of the lines we have drawn so far are called secant lines. A secant line touches a curve at least twice. Notice that as the distance between the two points of intersection decreases, Δt decreases. As Δt decreases, we get closer to an instantaneous rate.



All of the secant lines we have drawn so far have been drawn through point A and one other point. As the other point is moved closer to point A, the slope of the line approaches the instantaneous rate a A. Suppose that we moved the second point until it coincided with point A. Then the line would touch the curve at only one point. This is called a <u>tangent line</u> to a curve. The slope of this tangent line is the instantaneous rate at the corresponding time t. On the following page we have a magnified version of the graph near point A. We have shown two secant lines and one tangent line. Points P and Q are on the tangent line.

Notice that the tangent line lies between the two secant lines. This implies that the slope of the tangent line is between the slopes of the two secant lines. We have already calculated the slopes of the two secant lines. The results were as follows.

Secant Line AB: $m = -15$

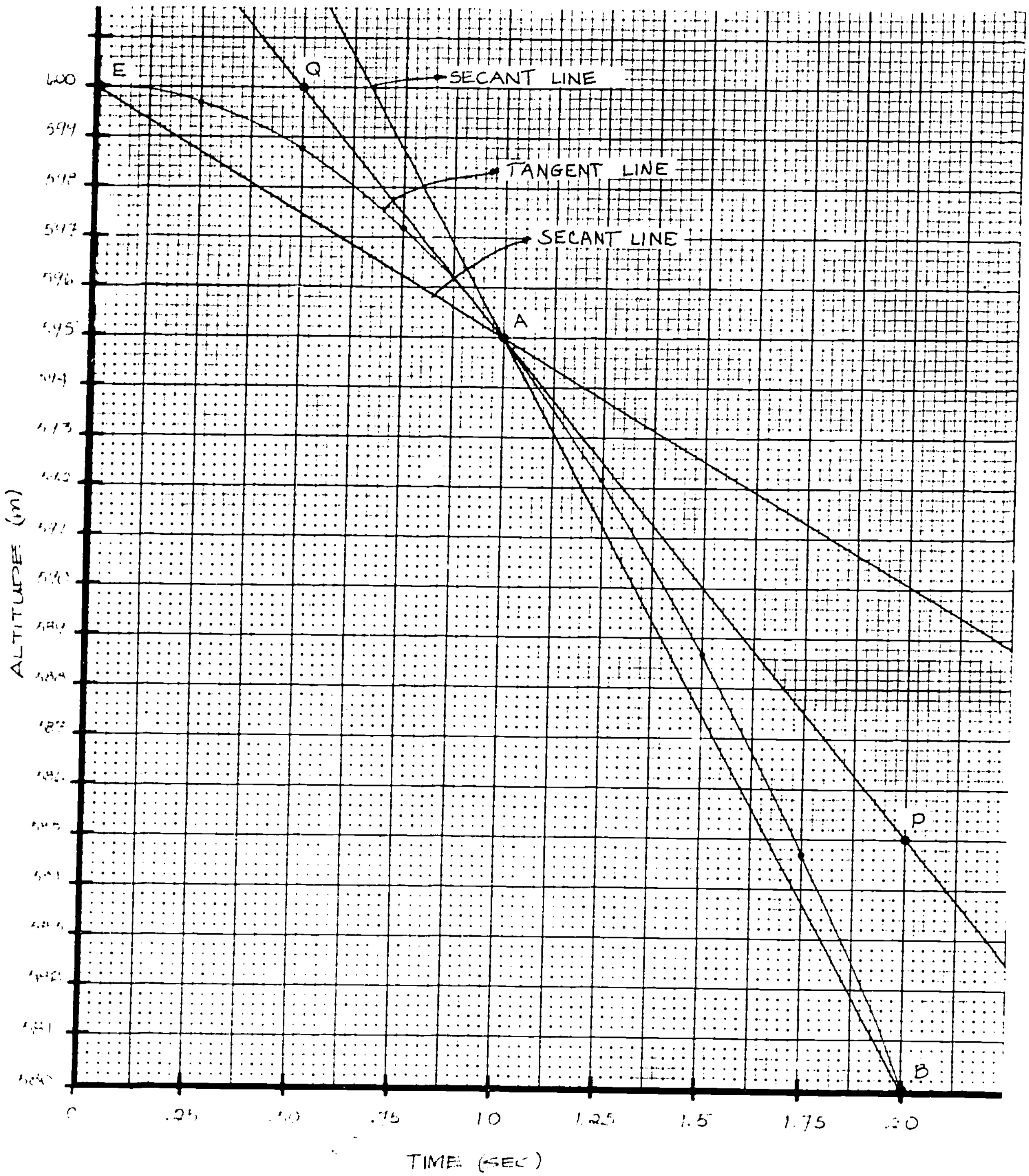
Secant Line AE: $m = -5$

By inspection of the graph we can determine the slope of the tangent line. Points P and Q both lie on the tangent line. Between the two points the rise, or Δy, is

$$\Delta y = 585 - 600$$
$$= -15$$

The run, or Δt, is

$$\Delta t = 2 - .5$$
$$= 1.5$$



The slope, or $\frac{\Delta y}{\Delta t}$, is

$$\frac{\Delta y}{\Delta t} = \frac{-15}{1.5}$$

$$= -10$$

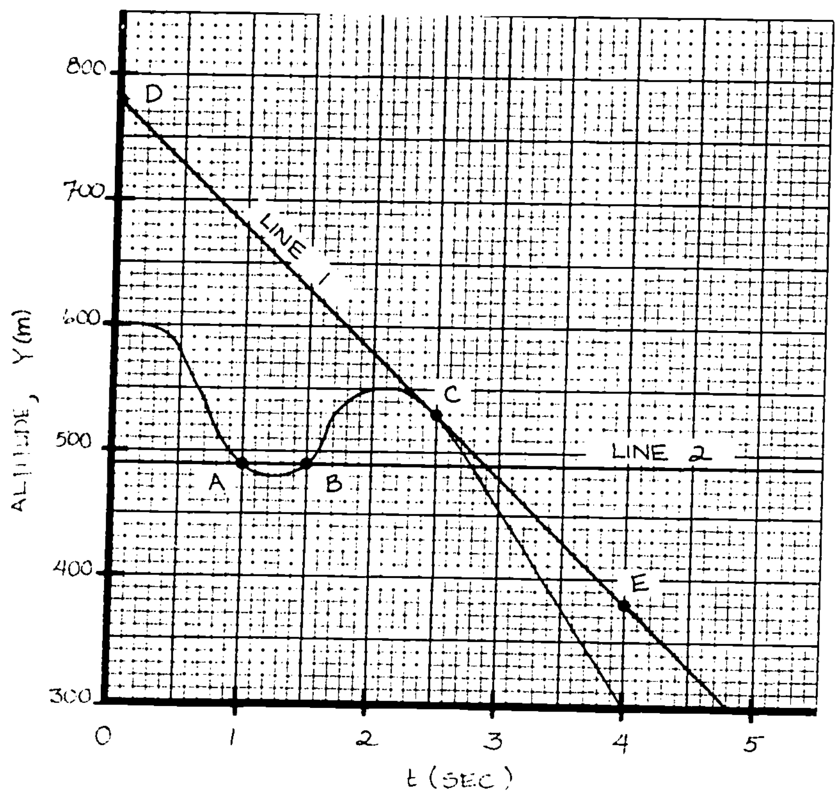
Therefore, the instantaneous rate at t = 1 sec is $-10\ \frac{m}{sec}$.

6-2 Tangent Lines, Secant Lines, etc.

The slope of a tangent line to a curve gives the instantaneous rate corresponding to the single point of contact. The slope of a secant line gives the average rate which corresponds to the two points of contact. We will use these ideas to determine rates from an inspection of the graphs of functions. For example, consider the next graph. It is a brief record of Elmo's experience while taking a flying lesson. Line 1 is a tangent line. The slope of this tangent line is the instantaneous rate which corresponds to the point of contact. The line touches the curve at t = 2.5 sec; therefore, the slope of the line at this point is the instantaneous rate at t = 2.5 sec. The slope of line 1 may be obtained from the graph. Points D and E are on the graph of the line 1. Δy between these points is (380 - 780), or -400 m. Δt is 4 seconds. Therefore

$$\frac{\Delta y}{\Delta t} = \frac{-400}{4}$$

$$= -100 \ \frac{m}{sec}$$

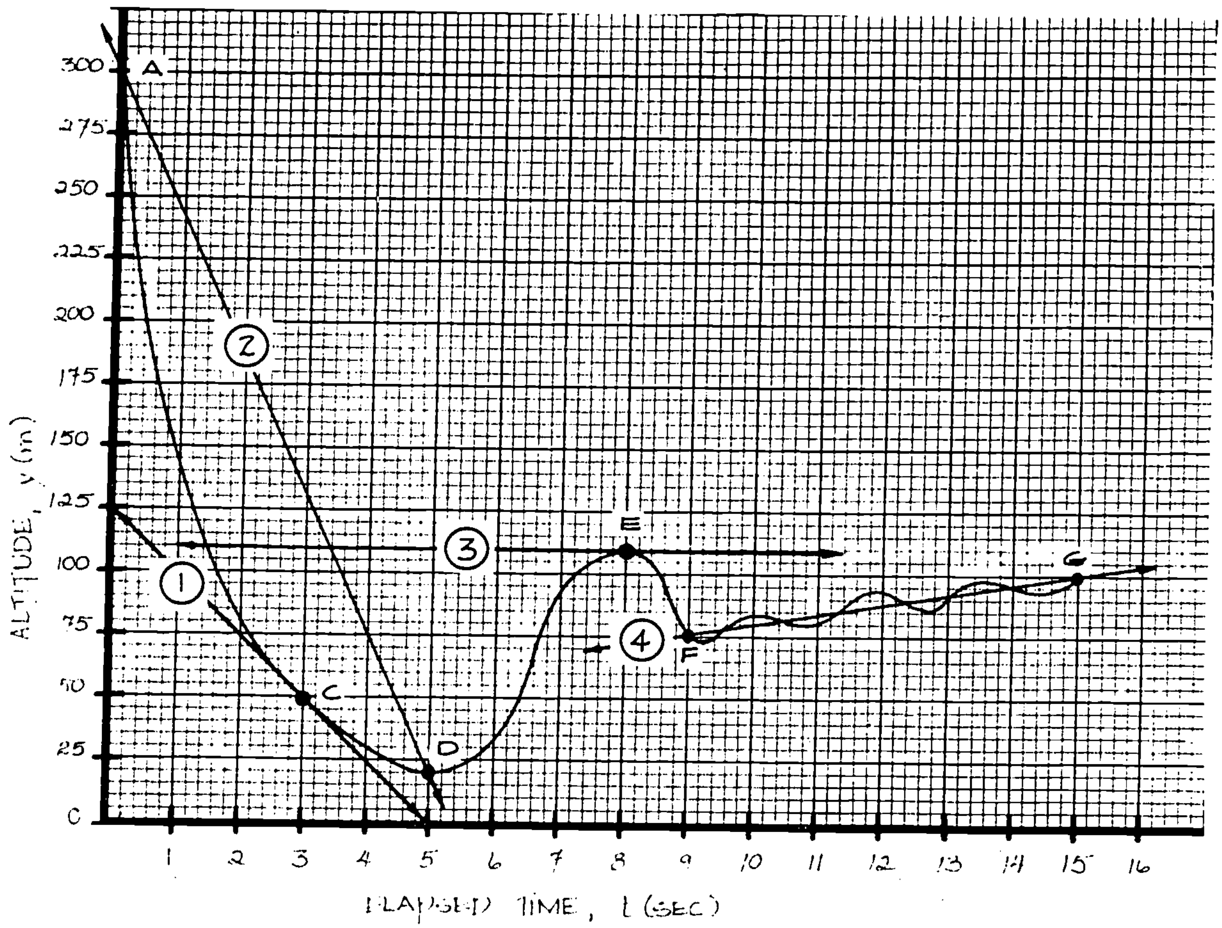


As we can see, Elmo was in quite a steep dive. We can also see from the graph that he went into an even steeper dive. After point C on the graph, the curve dives down even more steeply.

Line 2 is a secant line. The slope of this line is the average rate of climb between t = 1 sec and t = 1.5 sec. During this time interval Elmo's average rate was 0 $\frac{m}{sec}$.

PROBLEM SET 6:

Problems 1 through 11 refer to the graph below. It picks up the story of Elmo's flying lessons where the text left off.



1. Line 1 is a (tangent, secant) line. It touches the curve at point(s) ________. The slope of line 1 is the (instantaneous, average) rate at time t = ________.

2. Line 2 is a (tangent, secant) line. It touches the curve at point(s) ________. The slope of line 2 is the (instantaneous, average) rate in the interval t = ______ to t = ______.

3. The slope of line 3 is the instantaneous rate at t = ________.

4. The slope of line 3 is the average rate over the interval t = ______ to t = ______.

5. The slope of line 4 is the average rate between points ______ and ______ on the curve. What is the corresponding Δt?

6. What was Elmo's instantaneous rate at t = 3 seconds?

7. What was Elmo's average rate over the time interval t = 0 to t = 5 sec?

8. What was Elmo's average rate over the interval t = 1.4 to t = 8 sec?

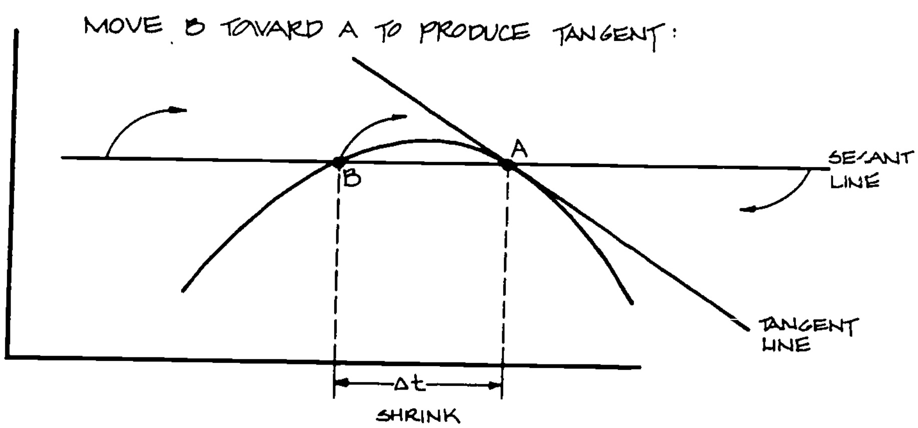
9. What was Elmo's instantaneous rate at $t = 8$ sec?

10. What was Elmo's average rate for $t = 9$ to $t = 15$ sec?

11. Elmo's rate was zero two times before $t = 9$ sec. When were these times?

SECTION 7: A LITERAL EXPRESSION FOR AVERAGE RATE

In the previous section we studied secant lines and tangent lines to curves. The slope of a secant line is the average rate which corresponds to the two points of contact. The slope of a tangent line is the instantaneous rate which corresponds to the single point of contact. We also saw that we could produce a tangent line by starting out with a secant line and moving one point along the curve until it coincided with the other point.



In this section we will duplicate algebraically what we have done so far numerically and graphically. We will start out by deriving a formula for the slope of a secant line near $t = 1$ sec.

STEP 1: Calculate y for $t = 1$ sec for the following equation. It describes Elmo's altitude as a function of time after leaving the airplane (from A to B on the graph is Section 5-1).

$$y = 600 - 5t^2$$

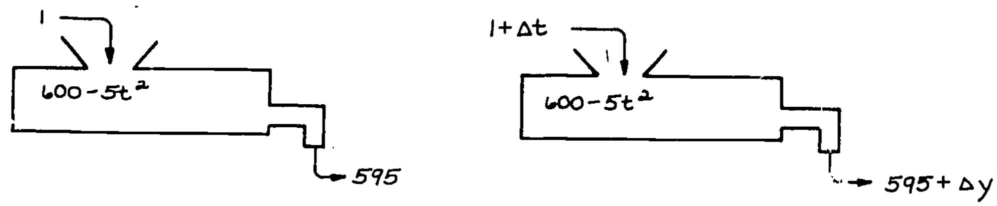
$$y = 600 - 5(1^2)$$

$$y = 600 - 5$$

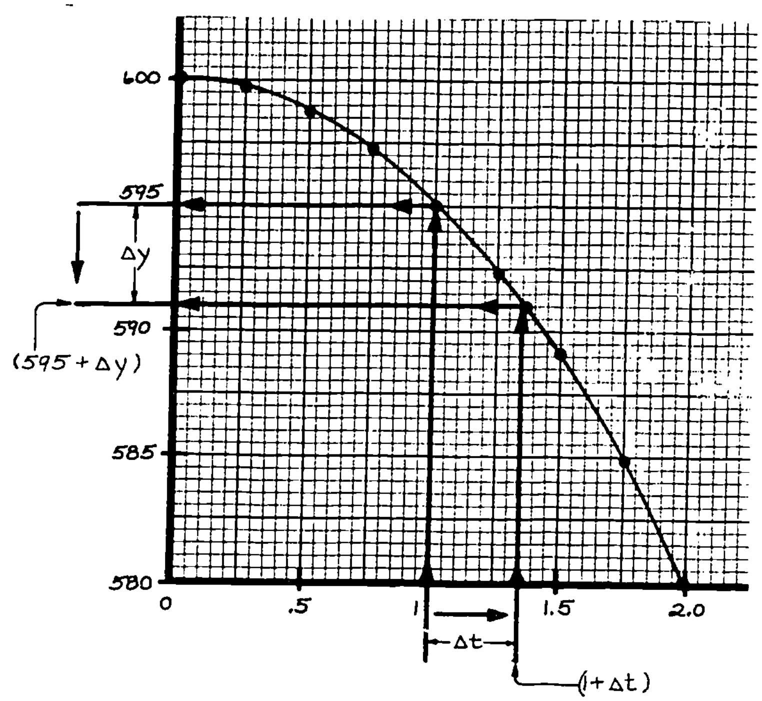
$$y = 595$$

STEP 2: Derive an expression for the change in y (or Δy) as a function of the change in t (or Δt).

Function machines will help us visualize what we are trying to do. On the left we drop in 1 and 595 pops out. On the right we drop in 1 + Δt and 595 + Δy pops out.



In other words, we are calling the new y the old one (595) plus any change in y (any Δy) which occurs as a result of Δt. This is the most important idea in this section. An examination of the following graph will give another point of view on this idea. When t = 1, y = 595. Next we see what happens when we change time a little from t = 1. We move a little to the right from t = 1 on the horizontal axis. We call the distance we move "Δt" or "the change in t." As we move to the right from t = 1 the corresponding value of y decreases. We call the size of the <u>change</u> in y "Δy" or "the change in y."



Finally, when t = 1 + Δt, y must be 595 + Δy. All that remains now is to substitute (1 + Δt) for t in the equation

$$y = 600 - 5t^2$$

and (595 + Δy) for y. This will put Δt on the right side and Δy on the left. (After a little algebraic simplification we will have an equation for Δy in terms of Δt.) Following through, let t = 1 + Δt and y = 595 + Δy. Then

$$595 + \Delta y = 600 - 5(1 + \Delta t)^2$$

After squaring the $(1 + \Delta t)$ term, we get

$$595 + \Delta y = 600 - 5[1 + 2\Delta t + (\Delta t)^2]$$

When the brackets are removed we get

$$595 + \Delta y = 600 - 5 - 10\Delta t - 5(\Delta t)^2$$

which reduces to

$$595 + \Delta y = 595 - 10\Delta t - 5(\Delta t)^2$$

Notice that there is a 595 term on both sides of the equation. When this term is subtracted from both sides, the expression becomes

$$\Delta y = -10\Delta t - 5(\Delta t)^2$$

We now have an explicit equation for the change in y (or Δy) in terms of the change in t (or Δt).

STEP 3: Derive an expression for the average rate (or $\frac{\Delta y}{\Delta t}$) as a function of the change in time (or Δt).

We start with the expression we derived in Step 2,

$$\Delta y = -10\Delta t - 5(\Delta t)^2$$

and notice that if both sides are divided by Δt we will have the desired formula.

$$\frac{\Delta y}{\Delta t} = \frac{-10\Delta t - 5(\Delta t)^2}{\Delta t}$$

$$= \frac{-10\Delta t}{\Delta t} - \frac{5(\Delta t)^2}{\Delta t}$$

$$= -10 - 5\Delta t$$

STEP 4: Try to figure out what will happen when $\Delta t = 0$.

We can do this by substituting $\Delta t = 0$ into the result of Step 3 and recalling that when $\Delta t = 0$, $\Delta y = 0$ too.

$$\frac{\Delta y}{\Delta t} = -10 - 5\Delta t$$

$$\frac{0}{0} = -10$$

As we have previously discussed, the quotient $\frac{0}{0}$ is indeterminate or, in other words, useless. The right side is better behaved. When $\Delta t = 0$, the right side is -10. And, we are finished. We have shown algebraically that the instantaneous rate at $t = 1$ sec is $-10\ \frac{m}{sec}$.

Not surprisingly this result agrees with the graphical results of the previous section. However, you should recognize the advantages of the analytical method described in this section. A graphical determination of the slope of a tangent line leaves a great margin for error. The analytical method is exact. It also bypasses the need to graph the function.

PROBLEM SET 7:

1. Show all work for this problem. $y = 600 - 5t^2$
   a. Calculate y for t = 3 sec.
   b. Derive an expression for Δy when $t = 3 + \Delta t$ and $y = 555 + \Delta y$.
   c. Derive an explicit equation for $\frac{\Delta y}{\Delta t}$ near t = 3.
   d. Determine the value of this expression when both $\Delta y = 0$ and $\Delta t = 0$.

2. Calculate the instantaneous rate when t = 2 sec and $y = 600 - 5t^2$.

3. Calculate the instantaneous rate when t = 4 sec and $y = 600 - 5t^2$.

4. Calculate the instantaneous rate when t = 3 sec and $y = 2000 - 16t^2$. In this equation y is altitude in feet and t is time in seconds.

SECTION 8: INSTANTANEOUS RATES AND DERIVATIVES

8-1 Elmo Joins the Circus

When Elmo finally graduated from Pudworthy High School, he yearned for a life of adventure. He thought that his sky-diving experience would make him good paratrooper material. So he tried to join the army; but the army didn't want him. Elmo had flat feet as a result of all the parachute landings he had messed up. Then he tried the air force. He also had dreams of becoming a pilot. The air force was interested until they talked to Elmo's flight instructor. Then, curiously, they lost all enthusiasm for signing up Elmo.

Then Elmo tried the circus. This time he had more luck. A position had just opened that was right up Elmo's alley. They told him that if he had come around just one day earlier he would have missed out. But luckily for Elmo the previous job holder had met with an unfortunate accident and would no longer be able to perform in his previous capacity. They mumbled something about, "Broken neck, tsk, tsk," and hurriedly went on to say that this was Elmo's big chance to be a star, make a lot of money, meet interesting people, travel around a lot and so forth.

At some point they pulled out a contract which Elmo signed with alacrity. But he still didn't have a clear idea of what it was that he had agreed to do. It didn't seem to matter, though. All day long friendly circus people escorted him around the grounds, introducing him to other circus people and involving him in circus-type chores like tent erecting and cage cleaning.

Finally, the hour of the big show arrived. They put him in a shiny silver suit with a shiny silver cape, and put a shiny silver football helmet on his head. Elmo felt quite grand. He just knew that he was going to be involved in something very spectacular. His only regret was that his old high school classmates couldn't see him now. They wouldn't laugh at him now that he was a big star.

Elmo still wasn't sure what his role was to be, but was too embarrassed to ask. How would it look if a star asked, "Uh,..., could anybody tell me what it is that I'm supposed to do?" Besides, these circus people seemed very nice. He didn't think that they'd involve him in anything dangerous.

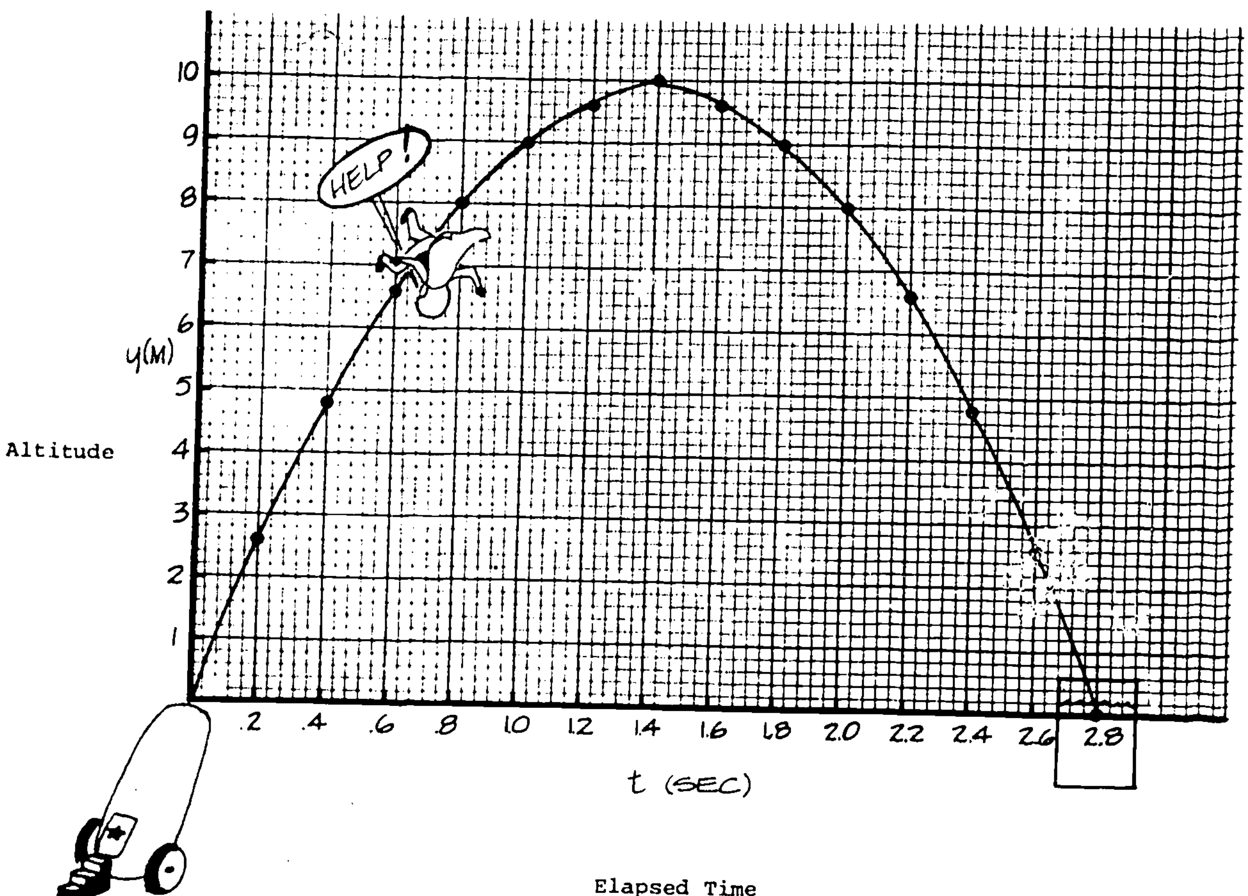
Soon they asked Elmo to climb into his "waiting room." Elmo found the waiting room to be quite confining and dark. It seemed to Elmo that it was a lot like a very tight-fitting cylinder. Elmo waited a short while and then something went "BLAM" and Elmo began to move very quickly. Suddenly it was very bright and Elmo realized that he was sailing through space. He had just enough time to scream "HELP!" before he landed with a huge splash in a pool of water.

As hastily as Elmo could, he shucked his silver suit and said "CENSORED" to the circus shifties that had suckered him into becoming for one night "The Magnificient Elmo, the Human Cannon Ball."

8-2 A More General Formula for Instantaneous Rate at Time t

The graph below describes Elmo's experience as a human cannon ball. On the vertical axis we have Elmo's altitude in meters. On the horizontal axis we have elapsed time in seconds. The curve is a graph of the equation $y = -5t^2 + 14t$. We will be interested in Elmo's instantaneous rate at time t. Since the vertical axis is Elmo's altitude in meters, the rate that we get will tell us how fast Elmo's altitude was changing. It will not tell us how fast Elmo was moving through the air. To see this, look at the curve at t = 1.4 sec. The slope of the tangent line there is 0. In other words, at that instant he was not going up or down. However, he was flying forward.

Thus far, we have answered questions like, "How fast was Elmo going up at t = 1.2 seconds?" This section will demonstrate how to answer the more general question,



"How fast was Elmo going up at $\underline{t}$ seconds?" Our goal here is to demonstrate a procedure that will yield a formula that will tell us the rate at any time t. The procedure will be directly analogous to the one we demonstrated in the previous section; however, there will be fewer numbers and more letters.

Elmo's altitude at time t is given by the equation

$$y = -5t^2 + 14t$$

Elmo's altitude $(y + \Delta y)$ at time $t + \Delta t$ is given by the expression

$$y + \Delta y = -5(t + \Delta t)^2 + 14(t + \Delta t)$$

Since $$(t + \Delta t)^2 = [t^2 + 2t\Delta t + (\Delta t)^2]$$

then $$y + \Delta y = -5[t^2 + 2t\Delta t + (\Delta t)^2] + 14(t + \Delta t)$$

When the brackets are removed, this becomes

$$y + \Delta y = -5t^2 - 10t\Delta t - 5(\Delta t)^2 + 14t + 14\Delta t$$

Now we want to subtract the equation

$$y = -5t^2 + 14t$$

from the expression with all the Δ's in it. When we do this we get

$$\begin{array}{rcl} y + \Delta y & = & -5t^2 - 10t\Delta t - 5(\Delta t)^2 + 14t + 14\Delta t \\ -(y & = & -5t^2 \qquad\qquad\qquad\qquad\quad + 14t \qquad\quad) \\ \hline \Delta y & = & \qquad - 10t\Delta t - 5(\Delta t)^2 \qquad\quad + 14\Delta t \end{array}$$

Now we have an explicit equation for Δy in terms of t and Δt. The main difference between this expression for Δy and ones obtained earlier is the presence of the variable t on the right side. This is the new feature.

Just as before, we divide both sides by Δt.

$$\frac{\Delta y}{\Delta t} = \frac{-10t\Delta t - 5(\Delta t)^2 + 14\Delta t}{\Delta t}$$

$$= \frac{-10t\Delta t}{\Delta t} - \frac{5(\Delta t)^2}{\Delta t} + \frac{14\Delta t}{\Delta t}$$

$$= -10t - 5\Delta t + 14$$

In order to find the instantaneous rate at t, we substitute $\Delta y = 0$ and $\Delta t = 0$ just as before.

$$\frac{0}{0} = -10t - 5(0) + 14$$

$$= -10t + 14$$

Again the left side is our mysterious friend $\frac{0}{0}$, while the right side is well behaved. We are done. The instantaneous rate for the function $y = 5t^2 + 14t$ at any time is given by the expression

$$(\text{instantaneous rate at time } t) = -10t + 14$$

Although the derivation of a general equation of this type may be more difficult to understand, it has its advantages. One of them is that we don't have to go through the lengthy procedure for each different value of t. We have done it once for all possible values of t. Whenever we need to know an instantaneous rate we merely substitute into the formula instead of dealing with all of those Δ's again.

For example, at what rate was Elmo going up at t = 0 sec?

$$\begin{aligned} m &= -10t + 14 \\ &= -10(0) + 14 \\ &= 14 \end{aligned}$$

Therefore, Elmo was going up at the rate of 14 $\frac{m}{sec}$ when he left the cannon. It may seem a little strange to have a rate other than 0 at t = 0, but keep in mind that Elmo was building up speed inside the barrel of the cannon <u>before</u> time zero.

8-3 One Last Example

<u>PROBLEM</u>:

How fast was Elmo going up at t = 2 sec?

(instantaneous rate at time t) = -10t + 14

$$t = 2$$

$$\begin{aligned} \text{rate} &= -10(2) + 14 \\ &= -20 + 14 \\ &= -6 \end{aligned}$$

Now, what does this -6 mean? Since the units for altitude are meters and the units for time are seconds it means that his rate was -6 meters per second.

The next problem is the meaning of the negative sign. What does going up at a rate of <u>negative</u> 6 meters per second imply? It means that Elmo was actually coming <u>down</u> at the rate of 6 meters per second.

8-4 A Notation Convention

When the quotient $\frac{\Delta y}{\Delta t}$ is evaluated at $\Delta t = 0$ it is customary to represent the instantaneous rate by the expression $\frac{dy}{dt}$. For example, Elmo's rate of change of altitude at time t is written

$$\frac{dy}{dt} = -10t + 14$$

and is read, "dee y dee t is -10t + 14."

In other words, instead of writing the expression, "instantaneous rate at time t," we write

$$\frac{dy}{dt}$$

The term that is used to describe $\frac{dy}{dt}$ is the word "derivative." In other words, -10t + 14 is the "derivative with respect to t of $y = 5t^2 + 14t$." The process of determining $\frac{dy}{dt}$ is called "taking the derivative." It is also called "differentiation," and to find $\frac{dy}{dt}$ is to "differentiate" y with respect to t. These new words and phrases all come from the area of study known as "the calculus."

Incidentally the term calculus is used by health professionals in a very different way--a gall stone or a kidney stone is often referred to as a calculus. As a matter of fact, both the mathematical term and the medical term stem from the Latin word "calculus," meaning pebble. In ancient times, strings of little stones, something like an abacus, were used for counting and calculating, and now calculus is used for the branch of mathematics that you are beginning.

PROBLEM SET 8:

For Problems 1 through 5 find the instantaneous rate at time t, or, in other words, $\frac{dy}{dt}$, for each of the given equations. In still other words, find the derivative of y with respect to t. Show your work.

1. $y = 29t^2 + 14t$
2. $y = \frac{t^2}{200} - 16t + 4$
3. $y = \sqrt{2}t^2 - 4t + 21$
4. $y = 38t^2 - 12t + 9$
5. $y = 4t^2 + \frac{t}{7} = 10$

6. Mathematicians refer to the process of finding $\frac{dy}{dt}$ as "taking the __________ with respect to t."

7. Use two English phrases to describe the work below. One phrase should use the word "rate" and the other should use the word "derivative."

$$y = 10t + 2$$
$$y + \Delta y = 10(t + \Delta t) + 2$$
$$= 10t + 10\Delta t + 2$$
$$y - y + \Delta y = (10t + 2) - (10t + 2) + 10\Delta t$$
$$\Delta y = 10\Delta t$$
$$\frac{\Delta y}{\Delta t} = 10\frac{\Delta t}{\Delta t}$$
$$= 10$$
$$\frac{dy}{dt} = 10$$

SECTION 9: TAKING THE DERIVATIVE OF A POLYNOMIAL

9-1 <u>What is $\frac{dy}{dt}$ for $y = kt^n$?</u>

The answer to this question is

$$\frac{dy}{dt} = knt^{n-1}$$

Section 9-3 explains how this result is obtained. However, those of you who do not like derivations may skip reading that section and go directly to the problem set.

Section 9-3 is only for those students who have some intellectual curiosity about why things are the way they are.

The following problem demonstrates the use of the basic formula.

PROBLEM:

Determine $\frac{dy}{dt}$ for $y = -5t^2$

SOLUTION:

The general formula states that when $y = kt^n$

$$\frac{dy}{dt} = knt^{n-1}$$

In our particular example

$$y = -5t^2$$

and we see that $k = -5$ and $n = 2$; therefore,

$$\frac{dy}{dt} = (-5)(2)t^{(2-1)}$$

or,

$$\frac{dy}{dt} = -10t^1$$

$$\frac{dy}{dt} = -10t$$

Unfortunately, we seldom encounter only one term on the right side. For example, Elmo's flight as a human cannonball was described by the equation

$$y = -5t^2 + 14t$$

which is the sum of two terms with t in them. Fortunately, this poses no major problems. We may apply the formula to each term on the right one at a time. The next sample demonstrates this procedure.

PROBLEM:

Find $\frac{dy}{dt}$ for $y = -5t^2 + 14t$

SOLUTION:

We rewrite the equation to show all exponents explicitly

$$y = -5t^2 + 14t^1$$

Now we apply the rule $\frac{dy}{dt} = nt^{n-1}$ to each term on the right, one at a time.

$$\frac{dy}{dt} = (-5)(2)t^{2-1} + (14)(1)t^{1-1}$$

$$= 10t^1 + 14t^0$$

Since $t^0 = 1$ the above equation reduces to

$$\frac{dy}{dt} = -10t + 14$$

which agrees with our earlier result.

9-2 Some Non-Obvious Extensions of the Formula

What happens when y is a constant? In other words, what is $\frac{dy}{dt}$ when $y = k$? We can find the answer to this question by using our powers of reasoning. Since y does not change, no matter how long we might wait, the rate of change must be zero. We can also answer the question by using our new formula. We remember that any number raised to the zeroth power is equal to 1. Specifically,

$$t^0 = 1$$

Now we can use this in the equation $y = k$, by observing that $y = kt^0$. Then we apply the differentiation rule to get

$$\begin{aligned}\frac{dy}{dt} &= k(0)t^{0-1} \\ &= k(0)t^{-1} \\ &= 0\end{aligned}$$

and we see that the formula gives us the same result that we expected by just thinking about rates of change.

Now we move on to a different sort of situation. We will show how to differentiate the function

$$y = \frac{k}{t}$$

where k is an arbitrary constant and t is some measure of time. Mathematicians say that y is inversely related to t. This terminology is based on the common-sense observation that as t gets larger, y gets smaller, hence the behavior of y is "inversely" related to t.

It is easy to give a couple of common qualitative examples of this sort of behavior. For example the selling price (y) of an automobile decreases as the car gets older (t increases). Another example. Your ability to expire air in a pulmonary funciton test (y) decreases the longer you try to expire air (t increases).

Now back to the mathematical business at hand--finding $\frac{dy}{dt}$ for $y = \frac{k}{t}$. It might not appear at first sight that we could apply our differentaition formula to this equation. The equation is not in the form $y = kt^n$. However, it is possible to write $y = \frac{k}{t}$ in this form.

First we remember from our work with scientific notation that

$$10^{-1} = \frac{1}{10^1} = .1$$

$$10^{-2} = \frac{1}{10^2} = .01$$

and in general

$$10^{-n} = \frac{1}{10^n} = \overbrace{.00\ldots01}^{n \text{ places}}$$

What is true for powers of 10 is true for powers of any number. In other words,

$$t^{-n} = \frac{1}{t^n}$$

We can apply this idea to the equation

$$y = \frac{k}{t}$$

We notice that this is equivalent to

$$y = k(\frac{1}{t})$$

and since $t^{-1} = \frac{1}{t}$,

$$y = kt^{-1}$$

Now the equation is in the $y = kt^n$ form where $n = -1$. We apply the differentiation rule to get

$$\begin{aligned} y &= k(-1)t^{-1-1} \\ &= -kt^{-2} \\ &= -\frac{k}{t^2} \end{aligned}$$

From this we see that we can find instantaneous rates for inverse relations. The following example shows how this process may be applied to negative powers of t greater than one.

PROBLEM:

$$y = \frac{3}{t} + \frac{10}{t^2} - \frac{6}{t^{10}}$$

Find $\frac{dy}{dt}$.

SOLUTION:

First we rewrite the equation so that all of the powers of t are removed from the denominators.

$$y = 3t^{-1} + 10t^{-2} - 6t^{-10}$$

Next we apply the rule $\frac{dy}{dt} = knt^{n-1}$ to each term.

$$\begin{aligned} \frac{dy}{dt} &= 3(-1)t^{-1-1} + 10(-2)t^{-2-1} - 6(-10)t^{-10-1} \\ &= -3t^{-2} - 20t^{-3} + 60t^{-11} \end{aligned}$$

We could leave the equation in this form or we could put the t's back in the denominators. This last possibility is shown below.

$$\frac{dy}{dt} = -\frac{3}{t^2} - \frac{20}{t^3} + \frac{60}{t^{11}}$$

9-3 Why Does the Formula Work?

Let's start out our explanation by simplifying things a little bit. We will let $k = 1$ in the equation $y = kt^n$ and take care of $k \neq 1$ later. So now we will focus our attention on the equation

$$y = t^n$$

and proceed as we have done earlier to derive an expression for $y + \Delta y$ as a function of $t + \Delta t$.

$$y + \Delta y = (t + \Delta t)^n$$

From our experience with the binomial theorem we know that the expansion of the term on the right will be of the form

$$(t + \Delta t)^n = t^n + nt^{n-1}\Delta t + \text{(terms containing } (\Delta t)^2 \text{ or higher powers of } \Delta t)$$

For reasons that will become obvious very shortly, we will be mainly interested in the first two terms of the expansion. For example, when n = 3 the first two terms of the expansions are

$$t^3 + 3t^{(3-1)}\Delta t = t^3 + 3t^2\,\Delta t$$

When n = 2 they are

$$t^2 + 2t^{(2-1)}\Delta t = t^2 + 2t\Delta t$$

And when n = 100 the first two terms will be

$$t^{100} + 100t^{99}\Delta t$$

Now we look at the rest of the terms. Whenever n is greater than one, there will be more than two terms in the expansion. All we need to know about the rest of them is that they will all contain powers of t that are two or greater. This is all we need to know because after we divide the right side by Δt and then substitute in $\Delta t = 0$, they will all disappear. More about this later. For convenience we will use the word "garbage" to refer to all of these terms.

So now we can write

$$y + \Delta y = (t + \Delta t)^n$$
$$= t^n + nt^{n-1}\Delta t + \text{garbage}$$

Since $y = t^n$ we can eliminate these terms from our expression.

$$y - y + \Delta y = t^n - t^n + nt^{n-1}\Delta t + \text{garbage}$$
$$\Delta y = nt^{n-1}\Delta t + \text{garbage}$$

To get an expression for the average rate we divide both sides by Δt.

$$\frac{\Delta y}{\Delta t} = \frac{nt^{n-1}\Delta t + \text{garbage}}{\Delta t}$$
$$= nt^{n-1} + \frac{\text{garbage}}{\Delta t} \quad \text{(= Average rate)}$$

Now it is time to look at the term $\frac{\text{garbage}}{\Delta t}$. We know from the binomial theorem that the garbage will have the form $(\text{constant})(\Delta t)^2 + \ldots + (\Delta t)^n$. When we divide the garbage by Δt we get $\frac{\text{garbage}}{\Delta t} = (\text{constant})\Delta t + \ldots + (\Delta t)^{n-1}$.

Now notice that when $\Delta t = 0$, the right side vanishes. Therefore when $\Delta t = 0$,

$$\frac{\text{garbage}}{\Delta t} = 0.$$

Now we return to our expression for average rate.

$$\frac{\Delta y}{\Delta t} = nt^{n-1} + \frac{\text{garbage}}{\Delta t}$$

When $\Delta t = 0$ we get

$$\frac{dy}{dt} = nt^{n-1} + 0$$

and we are finished with this part of the explanation.

To complete our explanation we must deal with the k term in the equation

$$y = kt^n$$

We proceed as before.

$$y + \Delta y = k(t + \Delta t)^n$$

$$= k(t^n + nt^{n-1}\Delta t + \text{garbage})$$

$$= kt^n + knt^{n-1}\Delta t + k(\text{garbage})$$

Since $y = kt^n$,

$$\Delta y = knt^{n-1}\Delta t + k(\text{garbage})$$

Dividing through by Δt we get

$$\frac{\Delta y}{\Delta t} = knt^{n-1} + \frac{k(\text{garbage})}{\Delta t}$$

and when $\Delta t = 0$

$$\frac{dy}{dt} = knt^{n-1}$$

and we are completely finished.

PROBLEM SET 9:

1. $\frac{dy}{dt}$ may be found for the equation $y = 16t^4$ by applying the relation $\frac{dy}{dt} = knt^{n-1}$ for the general equation $y = kt^n$.
   a. What is k for the equation $y = 16t^4$?
   b. What is n for the equation $y = 16t^4$?
   c. What is $\frac{dy}{dt}$ for $y = 16t^4$?
   d. What is $\frac{dy}{dt}$ when $t = \frac{1}{2}$?

Calculate $\frac{dy}{dt}$ for each of the following expressions.

2. $y = 4t^2 + 2t$
3. $y = \frac{t^{60}}{60} - t^3$
4. $y = 1 + 3t + 3t^2 + t^3$
5. $y = -4t^2 + \frac{t^{32}}{64}$
6. $y = -3t^{-2} + 2t^{-7}$
7. $y = at^{-6} + bt^6$

In Problems 8 through 12 determine $\frac{dy}{dt}$. The given expressions for y may be differentiated by first carrying out the indicated multiplications.

8. $y = t(1 + t)$
9. $y = (t - 1)^2$
10. $y = (t-1)(t-2)$
11. $y = t(t - \frac{1}{t} + \frac{1}{t^2})$
12. $y = \frac{1}{t}(t + t^2 - 1)$

13. Professor F. Lee Bitten had a lot of time on his hands. He had an unfortunate run-in with the law. Consequently he was spending a great deal of time in a state-operated institution of incarceration. To keep his mind sharp he studied the habits of some small cellmates, i.e., fleas. After years of intensive study he found that the distance (y) in meters of a flea from a central release point was related to the time (t) in seconds after release by the equation

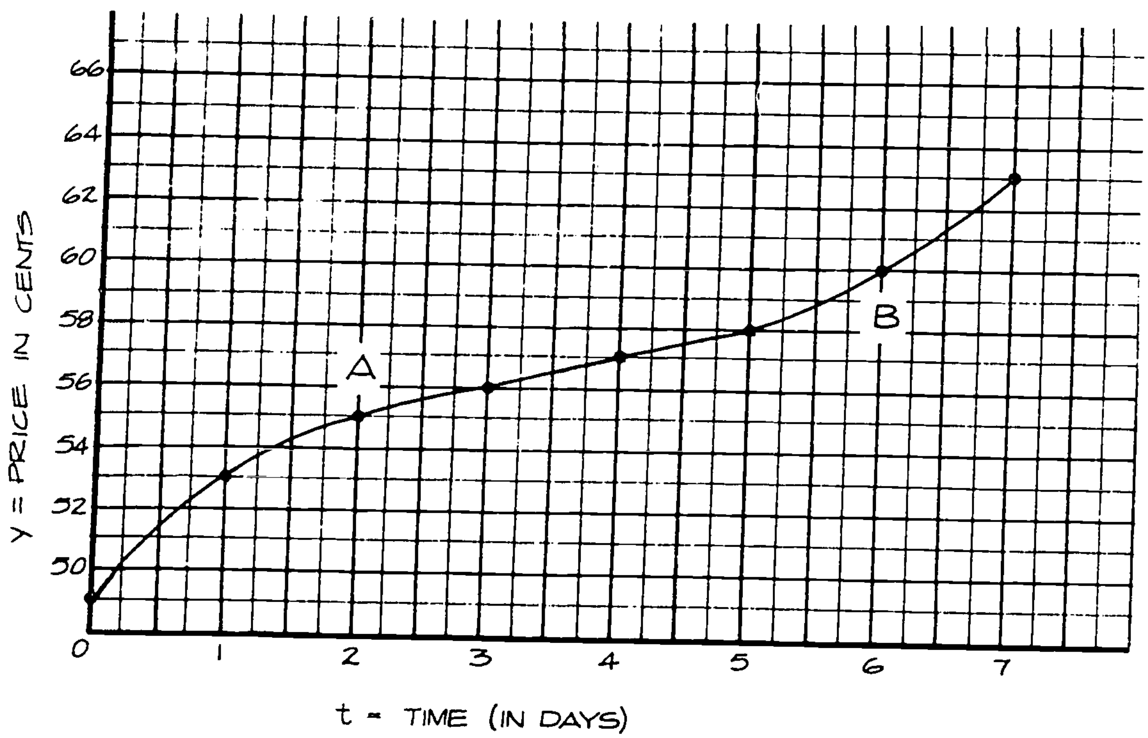
$$y = 3t^{16} - t^{14} + t^{11} - 3t^{7} + 4t^{3} - 6t^{2} + 2t^{-1} - \frac{2}{3}t^{-6} + 3t^{-9}$$

Show that the flea was moving away from the central release point at the rate of -1 meters per second, one second after release.

REVIEW PROBLEM SET 10:

Elmo went to the store on Monday to buy a quart bottle of Hootchie-Cola; the price was 40 cents. On each of the next seven days, Elmo came back for another bottle of cola; and each day the price was higher than before.

Here is a graph of price vs. time for a quart bottle of Hootchie-Cola.



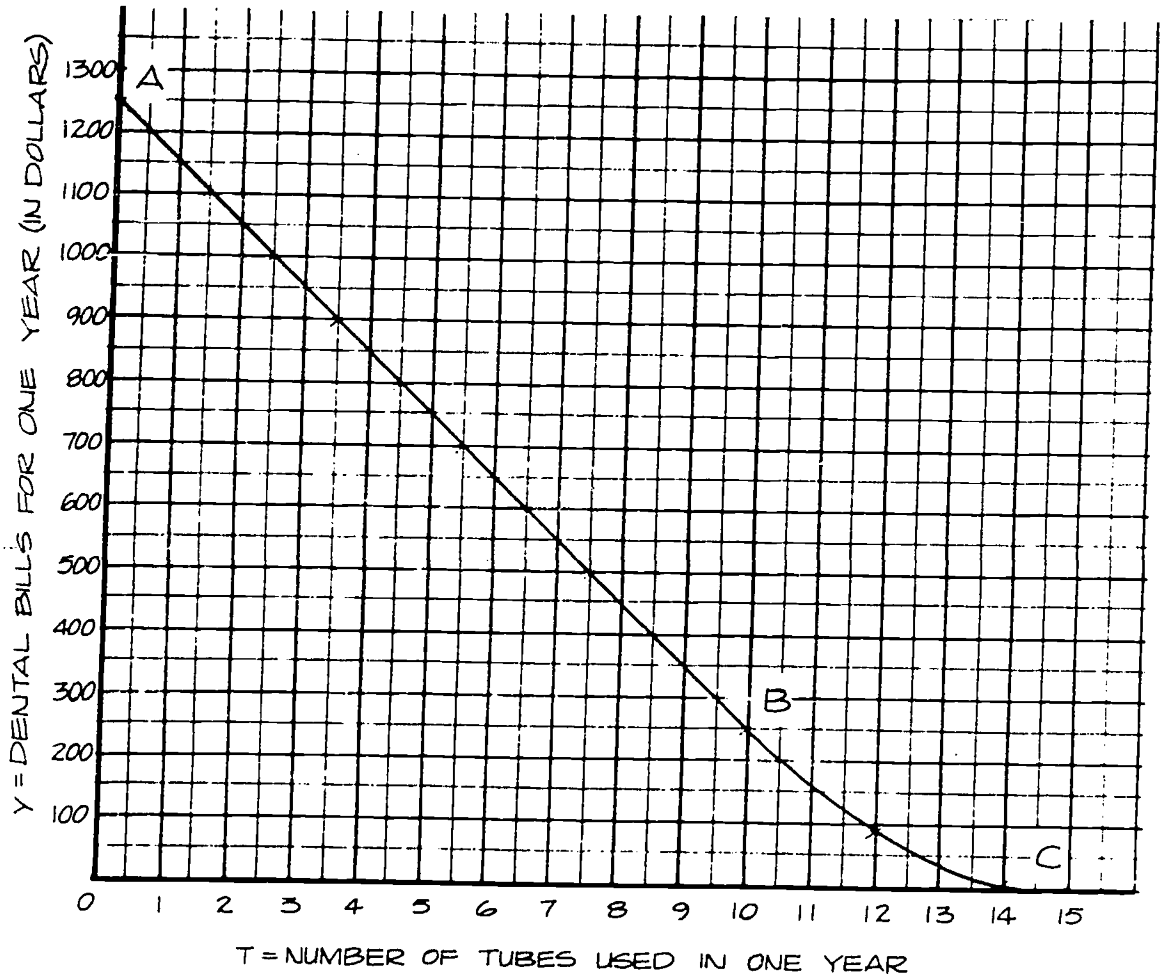
1. a. What was the average rate of change per day for the price, over the entire period?

b. What was the average rate of change per day for the period from the second day to the sixth day (Points A and B on the graph)?

Problems 2 through 5 deal with a fictitious relationship between dental costs and toothpaste.

The Fubar Toothpaste Company claims that you can save a lot of money on dental bills by using their toothpaste. In fact, they claim that the more you use, the more you save.

Below is a graph taken from one of their research reports.



The y values correspond to the amount of money a person spends on dental bills, for one year. The values indicate the number of tubes of Fubar toothpaste that a person used during the same year.

Notice that this graph consists of two parts:

a. from T = 0 to T = 10 (that is, from Point A to Point B), the graph is a straight line, with equation

$$y = -100T + 1250$$

b. from T = 10 to T = 15 (that is, from Point B to Point C), the graph is a parabola, with equation

$$y = 10T^2 - 300T + 2250$$

2. a. What is the average rate of change of y, between T = 2 and T = 7?

b. What is the average rate of change between T = 6 and T = 7?

3. a. Use the $\frac{\Delta y}{\Delta T}$ method to find the instantaneous rate for arbitrary T between T = 0 and T = 10.

b. Find the instantaneous rate for arbitrary T between T = 0 and T = 10, this time using the formula for $\frac{dy}{dT}$ when $y = kT^n$.

4. a. Find the instantaneous rate for arbitrary T between T = 10 and T = 15 using the $\frac{\Delta y}{\Delta T}$ method.

b. Find the instantaneous rate for arbitrary T between T = 10 and T = 15, this time using the formula for $\frac{dy}{dT}$ when $y = kT^n$.

The Point T = 10 (Point B on the graph) lies on both the linear and the parabolic portions of the graph.

5. a. Compute the instantaneous rate at Point B, considering this point as a part of the straight line. That is, use the formula $\frac{dy}{dT}$ for points on the line.

b. Compute the instantaneous rate at Point B, considering this point as a part of the parabola. That is, use the formula for $\frac{dy}{dT}$ for points on the parabola.

6. Differentiate the following functions. Use the formula for $\frac{dy}{dt}$ for $y = kt^n$.

a. $y = t^3 + \frac{1}{2}t^2 + 1$

b. $y = t^4 + t^2 + 2t$

c. $y = t^3 - 2t^2 + 3t - 1$

d. $y = \frac{1}{6}t^3 + \frac{1}{2}t^2 + t + 1$

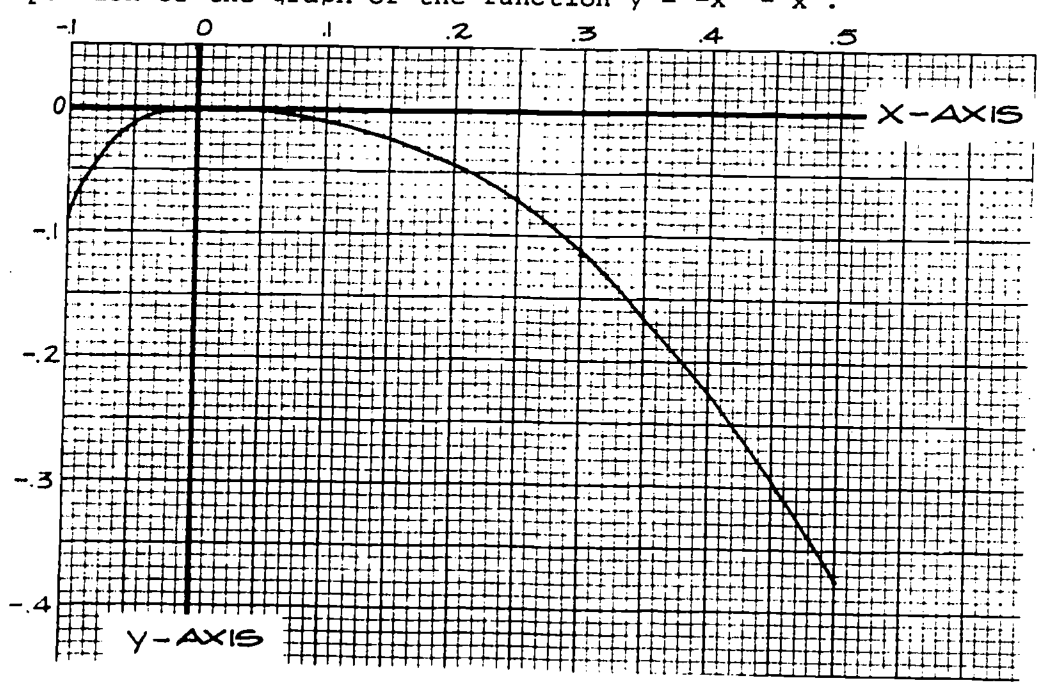
e. $y = \frac{1}{24}t^4 + \frac{1}{6}t^3 + \frac{1}{2}t^2 + t + 1$

f. $y = \frac{1}{120}t^5 + \frac{1}{24}t^4 + \frac{1}{6}t^3 + \frac{1}{2}t^2 + t + 1$

g. $y = \frac{1}{t^3} + 2t^2$

h. $y = t^{-4} + t^{-3} + t^5$

Below is a portion of the graph of the function $y = -x^3 - x^2$.



7. Show that the tangent to the curve at the point x = 0 has slope 0. That is, find the instantaneous rate at x = 0. (Hint: Find the derivative of $y = -x^3 - x^2$. Then substitute x = 0.)

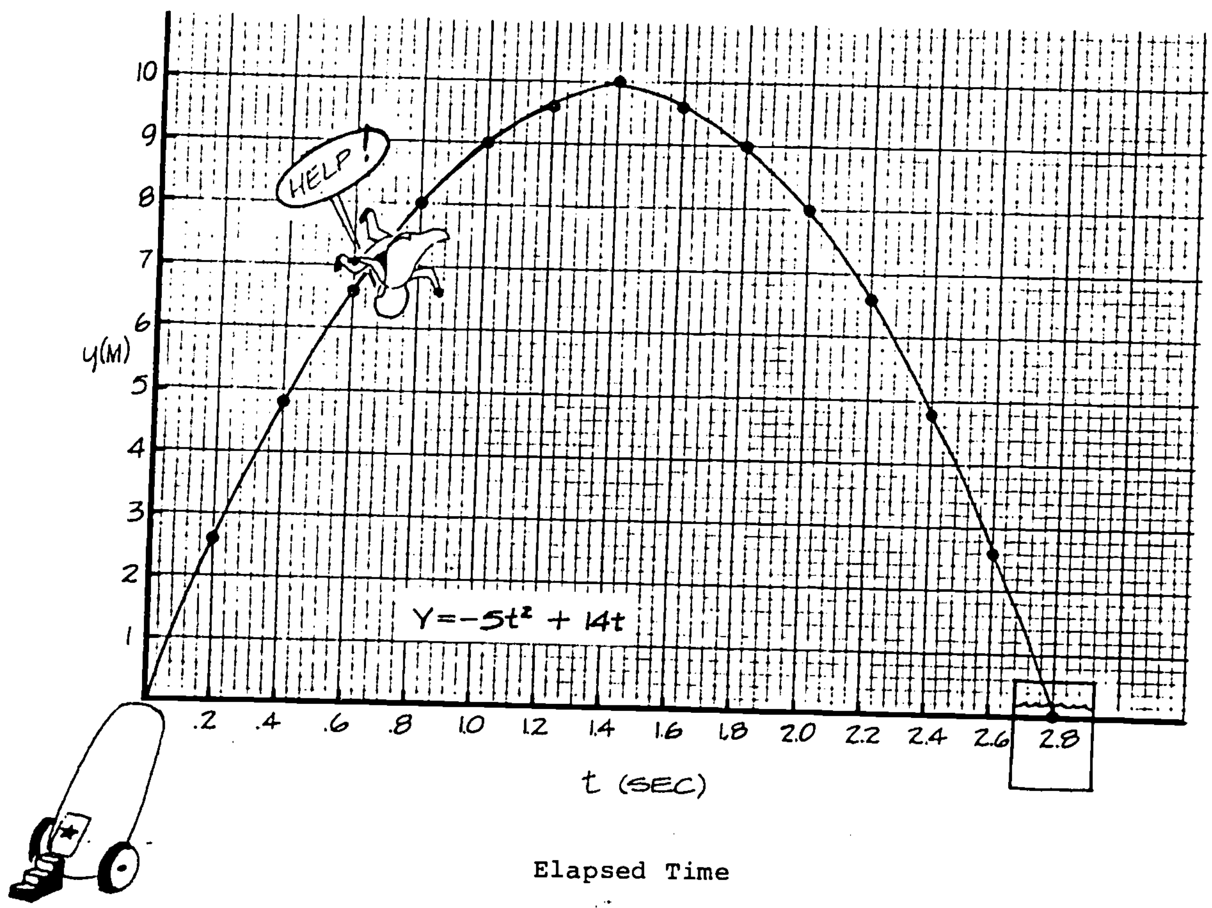
8. Find the slopes of the secant lines which cross the curve in the following pairs of points. That is, compute the average rates between the following pairs of x-values by substituting the x-values into the equation in Problem 7.

a. x = 0 and x = .5
b. x = 0 and x = .1
c. x = 0 and x = .05
d. x = -.05 and x = 0
e. x = -.1 and x = 0

SECTION 11: MAXIMA AND MINIMA

11-1 Peaks and Valleys

Now that we can differentiate polynomials we have all the tools necessary to solve another kind of problem. In order to introduce this new kind of problem we will return to Elmo's unhappy experience as a human cannonball and make a couple of additional observations. When he left the cannon he was going up. When he splashed down he was going down. Obviously, at some time in between these two times he stopped going up and started going down. Now look at the graph of Elmo's altitude as a function of time. At what time did Elmo reach his highest point? (In other words, when did he stop going up and start going down?)



Elapsed Time

He reached the peak of his flight at $t = 1.4$ sec. This is the time at which Elmo stopped going up and started going down. What was Elmo's instantaneous rate at this time? If he was neither going up nor down at this instant then his rate must have been zero. Another way to reach the same conclusion is to look at the graph and think about slopes of tangent lines. At the start of his flight the slope is positive. At $t = 1.4$ sec the tangent line is parallel to the x-axis. This means that the slope is zero at this point. Now we know that the slope of the tangent line for any time t is given by the derivative of the height function.

$$y = 5t^2 + 14t$$
$$\frac{dy}{dt} = -10t + 14$$

Now, suppose that we did not know beforehand that when $t = 1.4$, $\frac{dy}{dt} = 0$ and we wished to find out the instant when Elmo was highest. We would reason as follows. The slope of the tangent line must be zero at the peak of Elmo's flight. If we set $\frac{dy}{dt} = 0$, then we can solve the equation $0 = -10t + 14$ to find the instant when Elmo peaked out. Following through we have

$$0 = -10t + 14$$
$$-14 = -10t$$
$$\frac{-14}{-10} = t$$
$$1.4 = t$$

which naturally agrees with every other aspect of this overworked example.

We'll end this subsection by commenting on terminology. Occasionally mathematical language differs from what seems easy and natural. We are referring to the peaks and valleys of graphs. In this case, mathematicians prefer Latin to English. A peak is a maximum, peaks are maxima. A valley is a minimum and valleys are minima.

11-2 A Less Overworked Example

We will show that we can use the same line of reasoning described in the previous section to analyze a more complicated situation.

PROBLEM:

What values of t correspond to peaks and valleys (maxima and minima) of the function

$$y = 2t^3 + 6.6t^2 - 10.2t?$$

SOLUTION:

First of all we recognize that the question above is equivalent to asking, "For what values of t will $\frac{dy}{dt} = 0$?" Then we see that in order to answer this question we must first differentiate the function.

$$y = 2t^3 + 6.6t^2 - 19.2t$$
$$\frac{dy}{dt} = 2(3)t^2 + 6.6(2)t - 19.2$$
$$= 6t^2 + 13.2t - 19.2$$

Next we let $\frac{dy}{dt} = 0$ and solve the equation for t.

$$0 = 6t^2 + 13.2t - 19.2$$

We see with some little dismay that this is a quadratic equation. Of course we can solve it by a blunt application of the quadratic formula which we all immediately remember as

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

But b = 13.2 and there is a (b^2) term. Ugh! That's not the end of it. The (-4ac) term is (-4)(6)(-19.2). Few people enjoy the number cranking indicated here. Obviously it is high time to look for a shortcut of some sort. One shortcut that sometimes works is to divide the whole equation by the coefficient of t^2. In this case it is 6. If this works out nicely it has the effect of making all of the numbers involved here smaller. This generally tends to minimize calculation headaches. Following through, we have

$$0 = 6t^2 + 13.2t - 19.2$$

Dividing everything by six we get

$$\frac{0}{6} = \frac{6t^2}{6} + \frac{13.2t}{6} - \frac{19.2}{6}$$

$$0 = t^2 + 2.2t - 3.2$$

Now an application of the quadratic formula is less painful. The process is straightforward and the details should be familiar to you by now. We won't bore you with them here. The end result is

$$t = -1.1 \pm 2.1$$

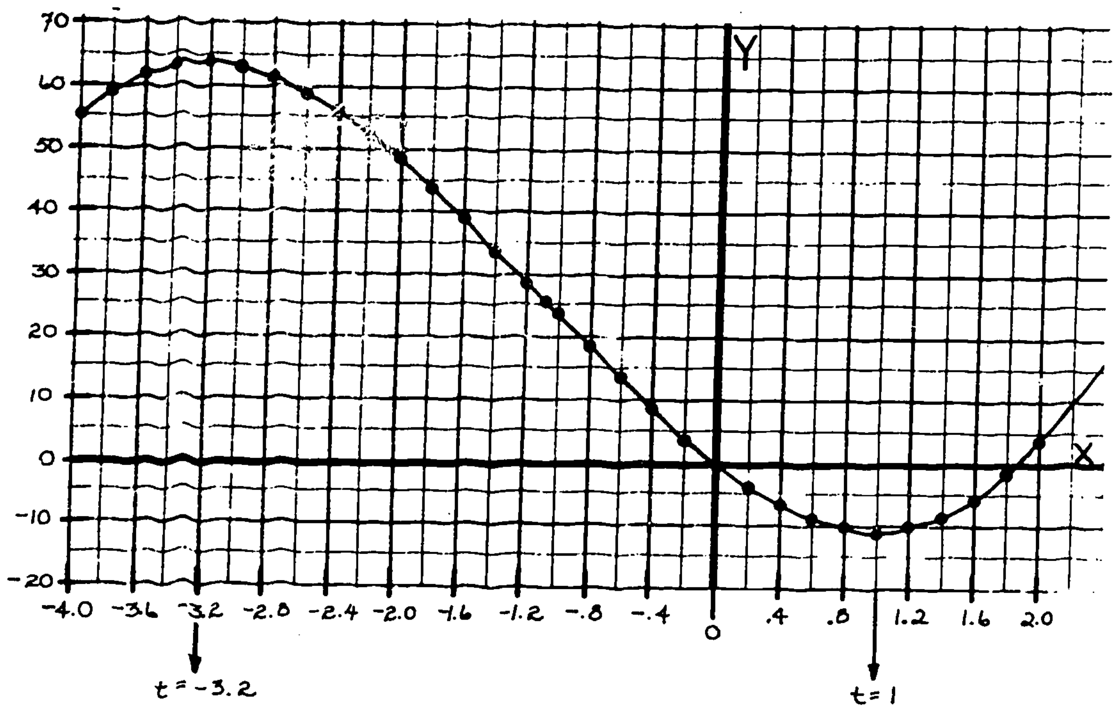
$$t = -3.2$$

and $$t = 1$$

These should be the t values which correspond to the maxima and minima of the function

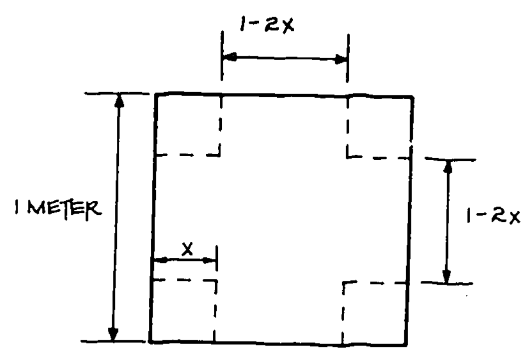
$$y = 2t^3 + 6.6t^2 - 19.2t$$

A graph of the function is shown on the following page. An inspection of the graph will confirm that there is indeed a valley that corresponds to t = 1 and a peak that corresponds to t = -3.2.



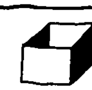
11-3 A Practical Example

PROBLEM: Suppose that we have a square piece of sheet metal 1 meter on a side that we wish to make into a container. We want to do it by cutting square pieces out of the corners as shown opposite.



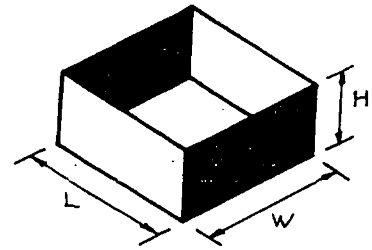
We wish to find out how big the little squares should be so that the container will have the maximum volume.

SOLUTION: We start out by making some rough observations. If x is small, then the resulting container will be very shallow and won't have much volume. On the other extreme we could make x very close to .5 m or half the length of the side. What would the volume of the container be then? The answer is that once again the volume would be small because the base would be very small. We can summarize the reasoning thus far is the table opposite.

X	V	CONTAINER
0	0	
SMALL	SMALL	
?	MAXIMUM	
ALMOST .5m	SMALL	
.5m	0	IMPOSSIBLE

We can see that the problem is shaping up as one of locating the peak volume for some value of x. In other words, we are looking for a maximum.

The second step on the road to the solution is to express the volume as a function of x. Recall that the volume of a rectangular solid is given by the length times the width times the height. Algebraically this is V = LWH.



Refer back to the diagram of the piece of sheet metal. The length and width will both be (1-2x). The height of the container will be simply x. Therefore,

$$V = LWH$$
$$= (1-2x)(1-2x)(x)$$

Now we have an expression for V in terms of x, but it is not in a form which we can differentiate. We will have to carry out the indicated multiplications first.

$$V = (1-2x)(1-2x)x$$
$$= (1 - 4x + 4x^2)x$$
$$= x - 4x^2 + 4x^3$$
$$= 4x^3 - 4x^2 + x$$

Now is the time to apply the method we introduced earlier. First we find the derivative

$$\frac{dV}{dx} = 12x^2 - 8x + 1$$

Second, we equate the derivative to zero and solve for x.

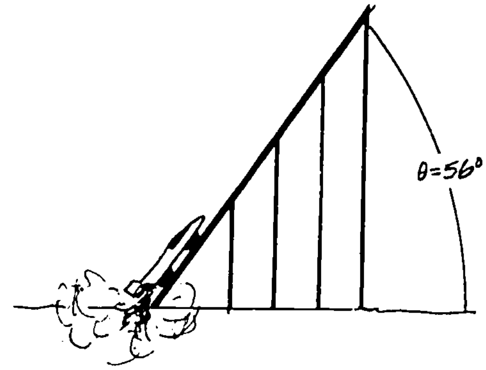
$$0 = 12x^2 - 8x + 1$$

This is a quadratic equation. The quadratic formula will give us x. (We will not divide everything by 12 because that would generate fractional coefficients and they are more difficult to calculate with than large whole numbers.

$$x = \frac{-(-8) \pm \sqrt{64 - 4(12)}}{24}$$
$$= \frac{8 \pm \sqrt{16}}{24}$$
$$= \frac{8 \pm 4}{24}$$
$$x = \frac{4}{24} \text{ or } x = \frac{12}{24}$$
$$x = \frac{1}{6} \text{ or } x = \frac{1}{2}$$

Since $x = \frac{1}{2}$ is impossible, $x = \frac{1}{6}$ is the answer. In other words, the container will have a maximum volume when the height of the container is $\frac{1}{6}$ meter and the length of the base is $[1 - 2(\frac{1}{6})]$ meter long or $\frac{2}{3}$ meter.

PROBLEM SET 11:



1. Greasy Weasel was an infamous stunt man. For a time he captured the imagination of the country with his proposed leap of the Salmon River canyon on his Heaven Cycle. The diagram opposite outlines how he proposed to attempt the jump. He planned to rocket off an inclined ramp at a speed of about 180 meters per second (400 mph). Greasy's engineers used vector analysis to determine that his vertical speed would be 150 m/sec and his horizontal speed 100 m/sec. The ramp had an inclination of 56°. It is possible to derive his height as a function of time after blastoff from this information. It is

$$h = 150t - 5t^2$$

where h = height in meters

t = time in seconds after blastoff.

a. Differentiate the height equation.

b. Equate $\frac{dh}{dt}$ to zero and solve for t. This is the number of seconds required for Greasy to reach his peak (maximum) altitude.

c. Substitute your answer to Part b into the height equation. This will tell you his peak altitude.

d. Greasy's technicians figured that his horizontal position at any time t would be given by the equation

$$d = 100t$$

where d = distance in meters

t = time in seconds after blastoff.

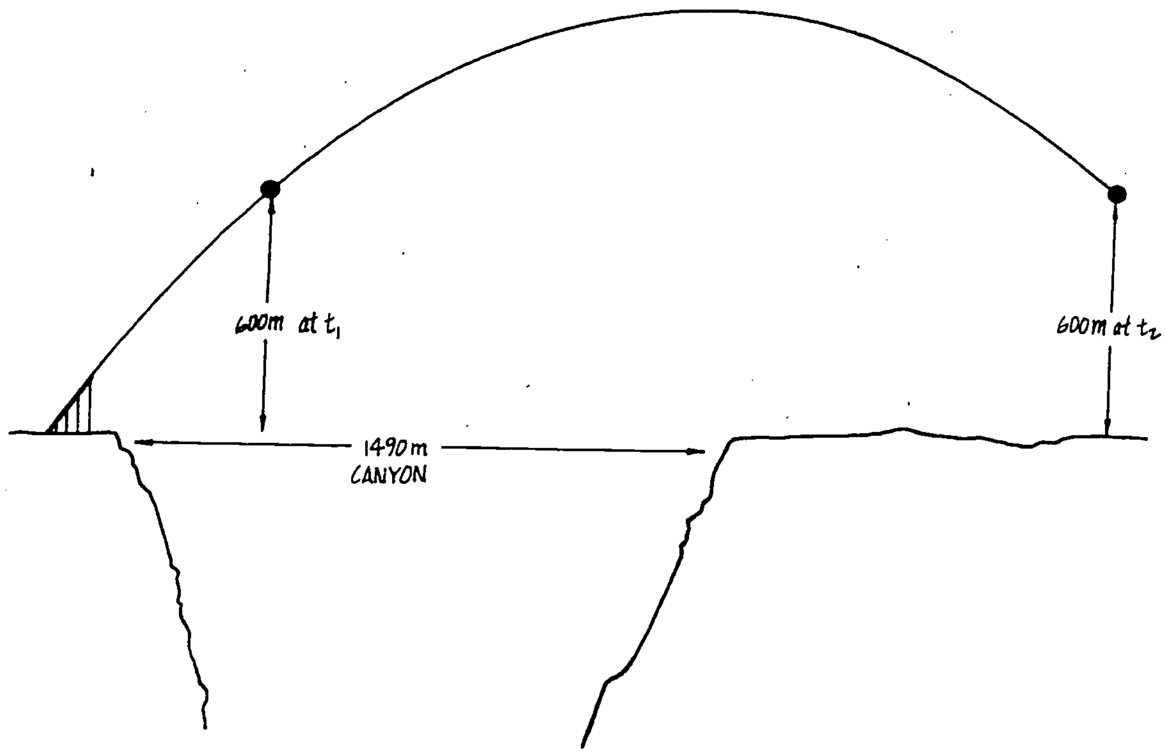
Substitute your answer to Part b into the above equation. This will tell you how far Mr. Weasel had traveled horizontally when he had reached his peak altitude.

e. Mr. Fearless, Mr. Weasel's nickname, had been told by his technicians that he shouldn't open his parachute lower than 600 m (about 2,000 ft.). The equation (*) below when solved for t will tell you the two times when Greasy had an altitude of 600 m. How many seconds after blastoff should Greasy pull the ripcord on his landing chute? State your answer to the nearest second. Diagramatically, we are finding the time t_2 that corresponds to the situation below.

$$600 = -5t^2 + 150t$$

$$0 = 5t^2 - 150t + 600$$

$$(*) \quad 0 = t^2 - 30t + 120$$



f. Substitute your answer to Part e into the equation d = 100t to find the horizontal distance that Greasy had traveled when his parachute had opened.

g. After Greasy's 'chute had opened he fell at the rate of 5 meters per second. By the time his 'chute was completely open his altitude was down to 500 meters. Therefore, his altitude at time t after the instant of deployment is given by the equation

$$h = \underbrace{500}_{\text{Initial height}} - \underbrace{5}_{\text{Falling rate}}t$$

where h = height in meters

t = time in seconds after full opening of 'chute

Substitute h = 0 into the above equation to find out how long it took Mr. Weasel to reach the ground.

h. After Greasy's 'chute opened he was at the mercy of the winds. Unfortunately, on the day of the jump there was a headwind of 11 meters per second (about 23 mph). Greasy had traveled 2600 m horizontally before the winds took control of his fate. Therefore, his horizontal distance after opening is given by the equation

$$d = 2600 - 11t$$

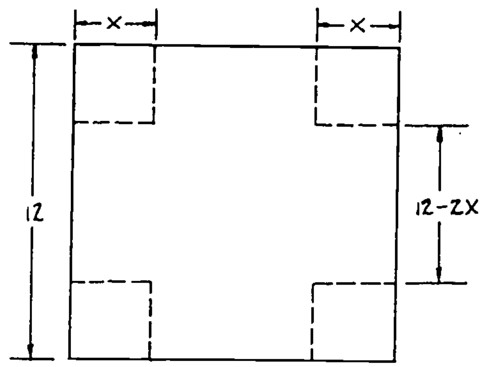
where d = horizontal distance from launching ramp (in meters)

t = time after full deployment of 'chute (in seconds)

Substitute your answer to Part g into this equation to find Greasy Weasel's horizontal position when his altitude was zero.

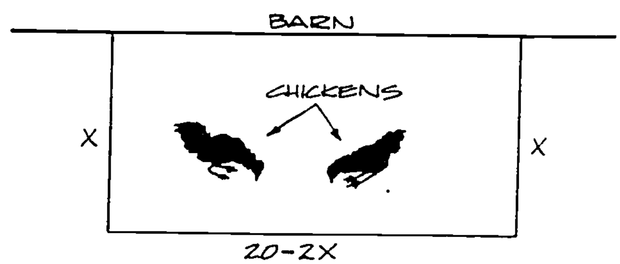
i. If your answer to Part h was less than 1490 m, then Greasy was blown back over the edge of the canyon. By how far did Mr. Fearless miss the edge of the canyon?

2. a. This problem is very similar to one described in the text. A square piece of sheet metal is to be folded to make a container. One side of the square is 12 meters long. Square pieces are to be cut out of each corner. How big should the pieces be so that the volume of the container is a maximum?



b. What is the volume of the resulting container?

3. Chester Chickeneater has 20 m of fencing. He wants to make a rectangular chicken pen with part of his barn forming one side.



a. Recall that the area of a rectangle is the length times the width. Write an expression for the area of the pen.

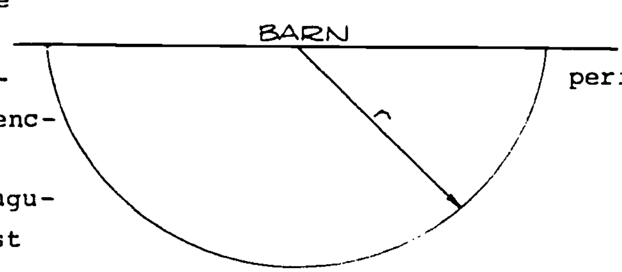
b. Differentiate the expression of Part a.

c. Equate $\frac{dA}{dx}$ to zero and solve for x.

d. Substitute the value of x found in Part c back into the area equation and calculate the corresponding area. This will tell you the maximum area for the rectangular pen.

e. Calculate the area of a semicircular pen made from 20 meters of fencing.

f. Should Chester build a rectangular or semicircular pen to get the most area?

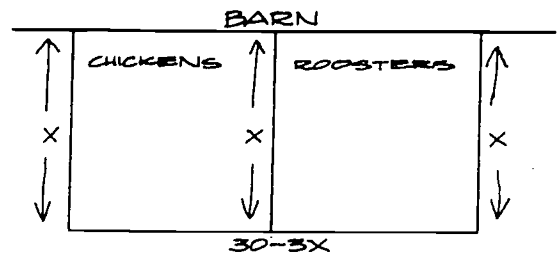


perimeter = π

20 = π

Area = $\frac{\pi}{2}$

4. Paul Tree is also a chicken rancher. He too wants to build a rectangular chicken pen with his barn forming one side. He has 30 m of fencing and he wants to build a partition between the hens and roosters to prevent random breeding. The pen is diagramed opposite.



a. What should x be so that the area is a maximum?

b. What is tne maximum area?

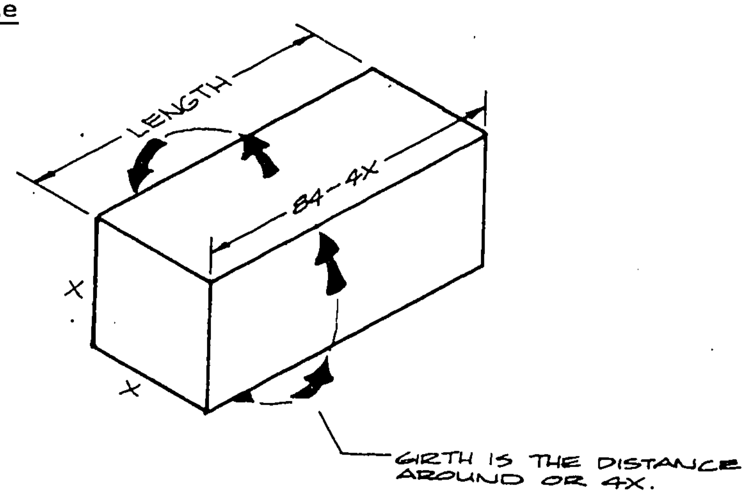
5. Parcel Post regulations state that the combined length and girth of a package cannot exceed 84 inches for a package shipped inside the United States. (The girth is the distance around the package as shown below.)

a. Write an expression for the <u>volume</u> of the diagramed rectangular package with the square ends.

b. Differentiate the equation with respect to x.

c. Equate $\frac{dV}{dx}$ to zero and solve for x.

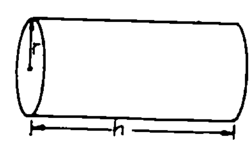
d. What is the maximum volume of such a package?



6. For packages shipped internationally there are different regulations. The length plus the girth may not exceed 72 inches. Calculate the dimensions of the maximum-volume vectangular package with square ends whose length plus girth does not exceed 72 inches.

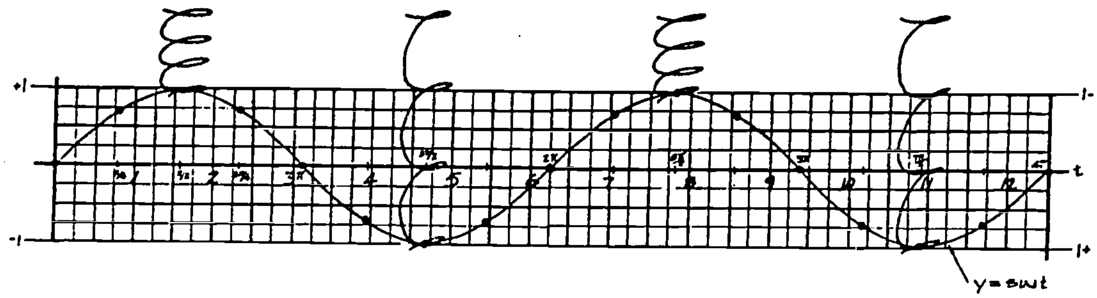
7. a. Calculate the radius and length of a cylindrical package with the maximum volume so that the length plus the girth does not exceed 72 inches.

b. Suppose that you wanted to ship some cloth from Guatemala to the United States. You could either fold it into a rectangular package with square ends or roll it into a cylindrical package. Which type of packing would permit you to ship the most material in one package? Justify your answer. You may use approximation $\pi \approx \frac{22}{7}$ in your answer.



volume $= \pi r^2 h$
girth $= 2\pi r$
$h = 72 - 2\pi r$

8. This problem requires us to refresh our memories concerning the two trigonometric functions sin θ and cos θ. Suppose that a perfect spring is stretched and released. It so happens that the displacement of the spring is a sine (or cosine) function of time. We have sketched a description of this situation on the following page.



Now that you have learned something about maxima and minima, you can determine some new facts about the behavior of the sine function.

a. Inspect the graph of sin t. List four values of t that correspond to peaks and valleys of sin t.

b. List four values of t that correspond to points at which the slope of the tangent line is zero.

c. What is the first time on the graph that corresponds to an instantaneous rate of change of position = 0? In other words, when did the spring first stop going up and start going down?

d. Inspect the graph near $t = \pi$. Estimate the slope of the curve at this point. This may be done by laying a ruler on the graph. Position the ruler so that it appears to be tangent to the curve. Then determine the slope of the ruler by estimating a rise-run pair.

e. Do the same thing for the curve at $t = 2\pi$.

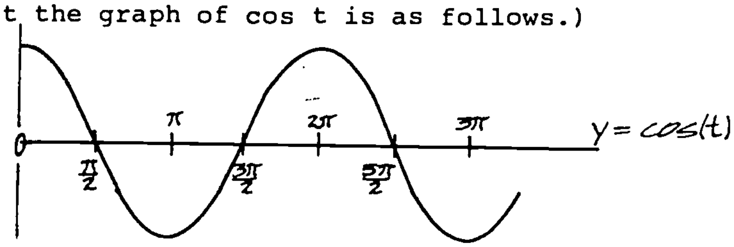
f. At what points on the graph does the slope have its greatest absolute value?

g. At what times is the spring moving fastest?

h. At what times is the spring moving slowest?

i. Fill in the table that follows.

(Recall that the graph of cos t is as follows.)



t	sin t	$\frac{d(\sin t)}{dt}$	cos t
0	0		
$\frac{\pi}{2}$	1		
π	0		
$\frac{3\pi}{2}$	-1		
2π	0		
$\frac{5\pi}{2}$	1		
3π	0		
$\frac{7\pi}{2}$	-1		

Problems 9 through 11 require some review of ideas relating to measures of central tendency. Recall that the mean and median are both measures of central tendency. The mean is the sum of a collection of numbers divided by the number of numbers. If you get scores of 85, 70, 90, 100 and 80 on five tests, your mean grade is $\frac{85 + 70 + 90 + 100 + 80}{5}$ or 85. The median is the "middle" value of a set of numbers. Half of the numbers will be greater than the median and half will be less. (In the example given, your median grade is 85.) There are some special cases that are more complicated than this, but this is the general idea.

9. We will start out with a simple example. Our collection of numbers is small. It is the set {1, 9}. We want to find a number, x, such that the sum of the squares of the deviations from x is a minimum. First we write an expression for the sum of the squares of the deviations.

$$s = (x - 1)^2 + (x - 9)^2$$

$$= (x^2 - 2x + 1) + (x^2 - 2\cdot 9x + 9^2)$$

Find x such that $\frac{ds}{dx} = 0$.

10. a. Find x such that the sum of the squares of the deviations from x is a minimum for the set of numbers {1, 3, 4, 6}.

b. Find the mean of the set of numbers {1, 3, 4, 6}.

11. In this problem we will attempt to generalize from the experience of the previous two problems. We will consider the set of numbers $n_1, n_2 \ldots n_z$ (in other words, a collection of z arbitrary numbers). What value of x will make the sum of the squares of the deviations from x a minimum?

a. First we write an expression for the sum.

$$s = (x - n_1)^2 + (x - n_2)^2 + \ldots + (x - n_z)^2$$

Show that

$$s = zx^2 - 2x\ (n_1 + n_2 + \ldots + n_z) + (n_1{}^2 + n_2{}^2 + \ldots n_z{}^2)$$

b. Show that when $x = \frac{(n_1 + n_2 + \ldots + n_z)}{z}$

the sum will be a minimum.

c. We have given a name to the expression in Part b. What is this name?

12. Why do tin cans have the shapes they have? Ima Goodwin asked this question of people on the street in her role as a newspaper reporter. Not too surprisingly she found that most people hadn't really thought about it before. However, when they thought about it a little bit they generally were able to come up with a reason. The reasons people came up with tended to fall into two broad categories, the subjective and the objective. Typical of the subjective reasons is the following one which was given by an artist. He said, "I think that manufacturers would want to choose a shape which is pleasing to the eye. They wouldn't want a shape which would be offensive. People wouldn't be inclined to pick it up. On the other hand, they would want a shape which would be eye-catching, a shape which would make the can stand out from other cans."

Typical of the objective replies is the one given by an engineer. She said, "Obviously, a manufacturer would want to make the cheapest container possible. One way to do this is to use the smallest amount of material which will contain the volume of product."

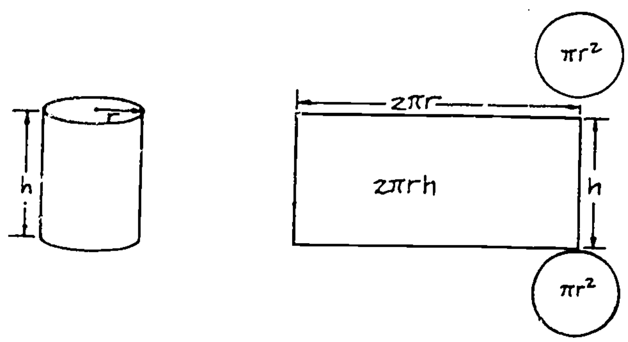
What motivates the canned-food people, aesthetics or economics? We can tackle this question by applying our "max-min" procedure to the material needed to make tin cans. By doing so we can determine the optimum shape of a can from the point of view of using the least material to contain the most product. If the shape of a can differs sharply from this optimum shape, then some other consideration was probably involved in deciding what shape the container was to have.

A typical tin can is known in the math trade as a "right circular cylinder." Everybody agrees the volume of such a beast is given by the formula

$$V = (\pi r^2)h$$

where πr^2 = area of base

h = height



The expression for the area is a little more complicated. On the right we have a picture of a tin can that we have worked on a little bit with a can opener and tin snips. We took a can opener to the ends and used the tin snips to slice the wall of the can from one end to the other. The wall is then laid out flat. This rectangle has a length equal to the circumference of the bottom. The width of the rectangle is equal to the height of the can. Consequently,

$$\begin{array}{ccccc} \text{Surface area of can} & = & \text{Surface area of wall} & + & \text{Surface areas of top and bottom} \\ A & = & 2\pi rh & + & 2\pi r^2 \end{array}$$

Notice that this equation has two variables on the right. We cannot apply our "max-min" procedure until we have area as a function of only one variable. We can arrange this by using the volume equation.

$$V = \pi r^2 h$$

and

$$\frac{V}{\pi r^2} = h$$

This may be substituted into the area equation.

$$A = 2\cancel{\pi}\cancel{r}\,\frac{V}{\cancel{\pi}r^{\cancel{2}}} + 2\pi r^2$$

$$A = \frac{2V}{r} + 2\pi r^2$$

$$A = 2Vr^{-1} + 2\pi r^2$$

Now we can differentiate A with respect to r while holding V constant.

a. Find $\frac{dA}{dr}$.

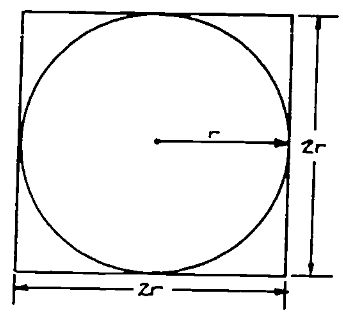
b. Equate $\frac{dA}{dr}$ to zero and solve for $\underline{r^3}$.

c. Substitute $\pi r^2 h$ for V in the equation found in Part b. Then, solve for r in terms of h.

d. Show that when the diameter is equal to the height then the amount of material in the can is a minimum for a given volume.

13. Oftentimes the material used in the can itself is not all the material actually used in the construction of the can. The tops and bottoms may be stamped from square pieces of metal. Consequently, the amount of sheet metal actually used in the construction of the can is given by the equation

$$A = 2(2r)^2 + 2\pi rh$$



Follow the procedure outlined in Problem 12 to show that when the diameter (D) is related to the height (h) by the equation

$$D = \frac{\pi}{4}\,h$$

$$\approx \frac{3}{4}\,h$$

then the amount of material actually consumed in the construction of the can is a minimum for any given volume.

The moral to this story is that if a tin can has a diameter, D, in the range $\frac{\pi}{4}\,g \leq D \leq h$ then it is efficient in its use of materials to contain its volume.

SECTION 12: A CRASH COURSE IN MOMENTUM

12-1 Momentum

In the Biomedical Science course you will be investigating trauma. Trauma is the branch of medicine that deals with smashed bodies and broken bones. In the United States automobile accidents are the leading cause of traumatic injuries. Since trauma is caused by human bodies colliding with inanimate objects like bridge abutments and cement trucks, we are going to have to learn a little physics about moving bodies. In math we will deal with the idea of momentum and rates of change of momentum. Momentum is simply the mass of an object multiplied by its velocity. It is a very central idea in the physics of collisions. Fortunately idea of momentum is intuitive. If you can answer the questions on the following ge correctly you already have a sense of momentum on a conscious level. On an unconscious level you are already using the concept of momentum. Running, walking, and standing all require a fundamental knowledge of momentum.

QUESTION 1: Two vehicles going 60 kph hit an overpass support on two different days. One was a VW bug. The other was a fully loaded cement truck. Which vehicle caused the most damage to the overpass support?

ANSWER: If you guessed the cement truck you were correct. The principle involved here is that the object with more mass will have more momentum when both objects have the same velocity.

QUESTION 2: Two cars of the same mass hit a bridge abutment on two different days. One was going 10 kph just before the collision. The other was going 60 kph just before its collision. In which automobile would you rather have been a passenger?

ANSWER: If you chose the slower moving car, your survival instincts are good. The principle involved here is that the faster an object is moving the more momentum it will have. In this case the object in question is your own body and it would have to have the same speed before collision that the different autos had.

We will conclude this subsection with a discussion of the algebraic statement of momentum.

$$p = mv$$

where "p" is the accepted abbreviation for momentum

m = mass

v = velocity

The letter "p" might seem like a strange choice for "momemtum," a word which starts with "m" and doesn't have "p" in it anywhere. However, it should be obvious that "m" would be a bad choice. "m" is already used as an abbreviation for two common physical quantities. It is used for both "meters" and "mass." To have it mean momentum as well would lead to mass confusion. On the other hand p has a powerful, explosive sound which is easy to associate with impulses and impacts, both of which are generally connected with changes in momentum.

Now let's look at the formula

$$p = mv$$

and see if it behaves the way we want it to. Recall the answer to Question 1. We concluded that if two vehicles were going the same speed, then the more massive one would have more momentum. We can easily show that the algebraic statement of momentum leads to this same conclusion.

Cement truck: mass = 23,000 kg

velocity = 60 kph

$p = (23,000)(60)$ kg·kph

VW Bug: mass = 1,000 kg
velocity = 60 kph
p = (1,000)(60) kg·kph

We can see that the cement truck had 23 times the momentum of the Bug.

Now recall the answer to Question 2. We concluded that if two objects had the same mass, then the faster of them would have more momentum. Imagine Ima Goodwin as a passenger of both cars.

Slow car: Ima's m = 55 kg
Ima's v = 10 kph
Ima's p = (55)(10) kg·kph

Fast car: Ima's m = 55 kg
Ima's v = 60 kph
Ima's p = (55)(60) kg·kph

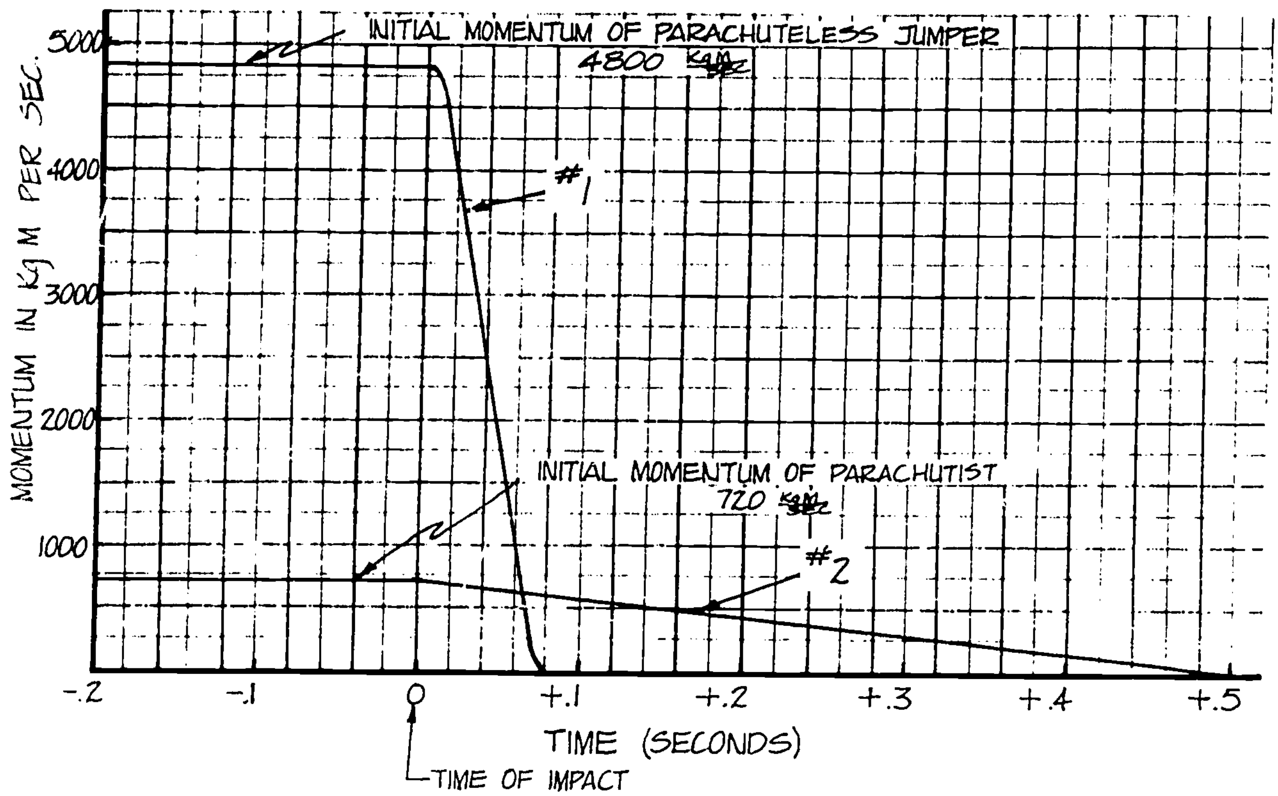
We can see that if Ima had been a passenger in the ill-fated fast car, her momentum would have been 6 times the momentum she would have had as a passenger in the slow car. And we see that the algebraic statement agrees in every way with our intuitive feelings about how things ought to be.

12-2 Rate of Change of Momentum

The rate of change of momentum is the major consideration in relation to what happens to a human body in a collision. The more rapid the rate of change the more likely that a severe injury will result. If the rate of change is slowed, a victim of a collision has an increased chance of survival. On a gross level this is what parachutes are for. They decrease the velocity of the falling body which in turn decreases the momentum with respect to the ground.

We have graphed momentum as a function of time for a parachutist and a parachute-less jumper. The time interval is near the moment of impact with the ground. Curve #1 describes the momentum of a 90-kg (200-lb) man falling at a terminal velocity of about ≈54 m/sec (120 mph). Curve #2 describes the change in momentum of an equally massive man under a parachute falling at about ≈8 m/sec (18 mph).

It is immediately obvious that the slope of Curve #1 is much steeper than Curve #2. We know by now, without calculating, that the absolute value of the rate of change of momentum is much greater for Curve #1. Upon closer inspection we notice that there are two reasons for this. First of all our unfortunate parachuteless jumper had more momentum to start with. Secondly, he took less time to stop. The reason for this may not be obvious. Suppose that both jumpers stopped in about the same length, that is, the length of their bodies. Since the parachuteless fellow was going faster, he covered this distance in less time. This effect may be sensed by performing a simple experiment. You will probably have to wait until you get home to do it unless you are in the habit of bringing a pillow with you for sleeping through classes. When you get a pillow, pound your palm into it at two



different speeds. One speed should be drastically slower than the other. Use the whole thickness of the pillow to stop your hand in. You should be able to feel the difference in time needed to bring your hand to a stop at the two speeds. At the faster speed it will take much less time to stop.

Finally, notice that the slopes are negative. This means that the momentums are decreasing. Physiologically, it is the absolute value of the rate of change of momentum that is important. Mathematically, the rate of change of momentum of jumper #1 was much less than that of jumper #2. However, the impact was much greater for #1. It is the absolute value of the rate, not its direction, that is important. In other words, the steeper the slope the more abrupt is the change in momentum and it is the abruptness that determines how damaging an impact is going to be.

12-3 Sample Rate of Change of Momentum Calculations

The first thing to do is to calculate the average rate of change of momentum for both jumpers. The unfortunate jumper #1 had an initial momentum of 4800 kg-m per second and a final momentum of 0, eight-hundredths of a second later; therefore,

$$\left|\frac{\Delta p}{\Delta t}\right| = \left|\frac{4800 \text{ kg-m per sec} - 0}{0 - .08 \text{ sec}}\right|$$

$$= \left|\frac{48 \times 10^2 \text{ kg-m per sec}}{-8 \times 10^{-2} \text{ sec}}\right|$$

$$= \left|-6 \times 10^4 \frac{\text{kg-m}}{\text{sec}^2}\right|$$

$$= 6 \times 10^4 \frac{\text{kg-m}}{\text{sec}^2}$$

Jumper #2 had an initial momentum of 720 $\frac{\text{kg-m}}{\text{sec}}$ and a final momentum of 0, five-tenths of a second later; therefore,

$$\left|\frac{\Delta p}{\Delta t}\right| = \left|\frac{720 \text{ kg-m per sec} - 0}{0 - .5 \text{ sec}}\right|$$
$$= \left|\frac{7.2 \times 10^2 \text{ kg-m per sec}}{-5 \times 10^{-1} \text{ sec}}\right|$$
$$= \left|-1.44 \times 10^3 \frac{\text{kg-m}}{\text{sec}^2}\right|$$
$$= 1.44 \times 10^3 \frac{\text{kg-m}}{\text{sec}^2}$$

You have no doubt noticed by now that the units of rate of change of momentum are much more complicated and unintuitive than any we have seen before.

Try not to think too much about what they mean, but use them as a check for your work. For example, if you have completed your calculations and the units are not (mass)·(length) per $(\text{time})^2$ then you have made an error.

Finally, you should be able to use dimensional algebra to transform the units of momentum. For example, in Questions 1 and 2 we used kg for mass and kph for velocity. You should be able to convert (kg-kilometers per hour) to (kg-meters per sec) or make any similar unit conversion.

Problem Set 12 is entitled "A Crash Course in Momentum." Several accidents are investigated in it. These accidents are based on reports of actual accidents, compiled by the U.S. Department of Transportation. The names have been changed to protect the victims.

PROBLEM SET 12--A CRASH COURSE IN MOMENTUM:

For Problems 1 through 4 use dimensional algebra to perform the indicated unit conversions. ($1 \text{ m} \approx 3.28 \text{ ft}$)

1. $100 \frac{\text{km}}{\text{hr}}$ to $\frac{\text{m}}{\text{sec}}$
2. $60 \frac{\text{mi}}{\text{hr}}$ to $\frac{\text{m}}{\text{sec}}$
3. $500 \frac{\text{kg-m}}{\text{hr}}$ to $\frac{\text{kg-m}}{\text{sec}}$
4. $1.38 \times 10^6 \frac{\text{kg-mi}}{\text{hr}}$ to $\frac{\text{kg-m}}{\text{sec}}$

5. Elmira Smith (age 34) was driving home one night (1:40 a.m.) in her compact car. She'd been at the local tavern lifting a few. She wasn't a lush, mind you. Oh, no. She thought of herself as a moderate, practical person. She wasn't a person who drank herself blind. Three drinks and she was through. And a prudent driver, too! She always fastened her seat belt, although not too tightly--sure didn't want to wrinkle her dress. However, it was late and she was tired and the booze really didn't help her stay alert. Furthermore, it was raining. Those oncoming headlights were really bright. The wet pavement seemed to be making the glare more intense. Yoicks! That guy left his brights on. While she was momentarily blinded she didn't see the curve to the left. She went off the road and hit a small tree. She was going about 15 m/sec (about 34 mph) when she hit the tree. The car came to a stop in .21 sec. Elmira's mass was 75 kg.

a. Calculate Elmira's momentum just before impact.

b. Calculate her absolute average rate of change of momentum over the entire time interval. $(|\frac{\Delta p}{\Delta t}|)$

Elmira survived the crash with only relatively minor injuries. She complained of chest pains. Presumably these were caused by her impact with the steering wheel. One factor which prevented more serious injury was the size of Δt. As you work through this problem set you will discover that .21 second is a long time for a vehicular impact. The reason Elmira's contact was longer was that the tree was small. The small-diameter tree penetrated farther into the car, giving it a longer distance in which to stop which resulted in a longer stopping time. The moral to this story is, "If you have to hit a tree, hit a little one."

6. Felix Bugdriver (25) was driving home from the local tavern. It was 3:00 a.m. He'd had a beer or two and he was sleepy--oh, so sleepy. It occurred to him that he shouldn't be driving when he was fighting off the z's. But he wasn't alarmed. He was too content and fatigued to be alarmed. Snore. Crash. Whoops. Those 70-cm diameter trees don't give like the pillows he should have hit. The collision changed the speed of Felix's 90-kg mass (about 198 lb) from 16 m/sec to zero m/sec in 14 csec. (A csec is one-hundredth of a second.)

a. Calculate Felix's momentum in (kg-m) per (sec).

b. Calculate Felix's absolute average rate of change of momentum in (kg-m) per (sec-csec).

c. Convert your answer to Part b into units of (kg-m) per (sec^2).

Felix survived the crash with serious injuries. He broke several bones in his chest and legs and he had a concussion. He wasn't wearing a seat belt. One factor which made Felix's injuries more serious than Elmira's was the diameter of the tree. Felix's was bigger than Elmira's. It stopped him in a shorter distance and hence a shorter length of time.

7. Busby Berzerkly and his drinking buddy Stewart Pid were driving home from the bar. Now Busby was one of those people who regularly drank himself blind. Acquaintances described him as "almost crazy." He had had several driving violations and been prohibited from driving or owning an automobile. This didn't stop mad Busby. He conned his landlady into registering his automobile and continued his bad boozing ways.

On this particular night it was really foggy. Not outside, but inside. Inside Busby's head, that is. A blood-alcohol concentration of .27% will fog up just about anybody's brain. Busby was no exception. Stew Pid had also been drinking, but his blood-alcohol level was only .07%, not even legally drunk (in most states legal drunkeness begins at .10%). So he couldn't blame the fact that he was riding with a madman on booze-blurred judgment.

a. Calculate Elmira's momentum just before impact.

b. Calculate her absolute average rate of change of momentum over the entire time interval. ($|\frac{\Delta p}{\Delta t}|$)

Elmira survived the crash with only relatively minor injuries. She complained of chest pains. Presumably these were caused by her impact with the steering wheel. One factor which prevented more serious injury was the size of Δt. As you work through this problem set you will discover that .21 second is a long time for a vehicular impact. The reason Elmira's contact was longer was that the tree was small. The small-diameter tree penetrated farther into the car, giving it a longer distance in which to stop which resulted in a longer stopping time. The moral to this story is, "If you have to hit a tree, hit a little one."

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a. Calculate Felix's momentum in (kg-m) per (sec).

b. Calculate Felix's absolute average rate of change of momentum in (kg-m) per (sec-csec).

c. Convert your answer to Part b into units of (kg-m) per (sec^2).

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On this particular night it was really foggy. Not outside, but inside. Inside Busby's head, that is. A blood-alcohol concentration of .27% will fog up just about anybody's brain. Busby was no exception. Stew Pid had also been drinking, but his blood-alcohol level was only .07%, not even legally drunk (in most states legal drunkeness begins at .10%). So he couldn't blame the fact that he was riding with a madman on booze-blurred judgment.

Authorities were able to determine from the length of skid marks, etc., that the speed of impact was about 15 m/sec (about 34 mph). The masses of the brothers Grease are listed in the following table.

	mass	weight	speed
Axel	72 kg	159 lb	$15\ \frac{m}{sec}$
Elbo	75 kg	165 lb	$15\ \frac{m}{sec}$
Easy	64 kg	141 lb	$15\ \frac{m}{sec}$

The impact was over in 8 csec.

a. Calculate the initial momentum for each passenger.

b. Calculate the absolute average rate of change of momentum for each passenger in $\frac{kg\text{-}m}{sec\text{-}csec}$.

c. Convert your answers to Part b to $\frac{kg\text{-}m}{sec^2}$.

9. Rick and Whit Less were brothers. They liked to have fun together. Their idea of fun was to score a couple of six-packs of malt liquor, drive up on top of the hill just outside of town, chug the booze and drive back into town. This was fun because it was exciting. It was exciting because it was illegal. It was illegal because it was dangerous. Read on, see how dangerous. Both Rick and Whit were under the legal drinking age. However Rick was old enough to drive. Therefore they were able to drive outside of town where they wouldn't get caught while they were putting away the juice.

One Friday night after a football game, Rick and Whit decided to have some fun. They took Ophelia Beese with them. On the way back down the hill the bald right front tire blew out. Rick over-corrected to the left. Unfortunately he collided with a bridge abutment. Both Whit Less and O. Beese died more or less instantly. Rick Less walked away.

From the length of the skid marks and the depth of penetration of the bridge abutment into the wrecked car, the police were able to estimate that the car had been going about 25 m/sec (56 mph) and that the impact had lasted about 12.5 scec. The masses and ages of the occupants are given in the following table.

	mass	age
Rick Less	60 kg	19
O. Beese	96 kg	34
Whit Less	64 kg	16

a. Calculate the initial momentum for each passenger.

b. Calculate the absolute average rate of change of momentum for each passenger in units of (kg-m) per (sec-csec).

c. Convert your answers to Part b to units of (kg-m) per (sec^2).

SECTION 13: INSTANTANEOUS RATES OF CHANGE OF MOMENTUM

In the previous section we learned that the rate of change of momentum is an important factor in collisions. You may already know or have guessed that the rate of change of momentum is actually force. You can perform a simple experiment to convince yourself of this fact. Clap your hands first gently and then hard. When you clap your hands hard, the impact of your two hands when they collide is much more strongly felt than when they are clapped gently. You should be able to sense the greater force of impact when they are clapped hard. From a physical point of view, this is equivalent to saying that the rate of change of momentum is greater when the hands are clapped hard.

There is a parallel between the hand clapping example and the parachute example of the previous section. The average rate of change of momentum is $\frac{\Delta p}{\Delta t}$. Δp is larger for a hard hand clap because the hands are moving faster. Δt will tend to be shorter for the hard hand clap again because the hands are moving faster. Both of these effects will tend to make the ratio $\frac{\Delta p}{\Delta t}$ greater for the hard hand clap than for the gentle one. You feel this greater rate of change of momentum as a momentary force. When we calculate an average rate of change of momentum we are calculating the average force exerted on the object during the time interval that is used in the calculation.

All of this so far is important to know, but it isn't quite enough. It is possible for all passengers in an automobile to experience approximately equal average rates of change in momentum. In other words Δp will be about the same and Δt will be the time it takes for the car to stop; therefore the $\frac{\Delta p}{\Delta t}$ ratios for the passengers will be about equal. We will see that the differing instantaneous rates are a major influence in causing different passengers to sustain injuries of widely differing seriousness. Seat belts, shoulder harnesses, air bags, padded dashes, and steering wheels are things that influence the pattern of instantaneous forces that occur during the course of an automobile collision.

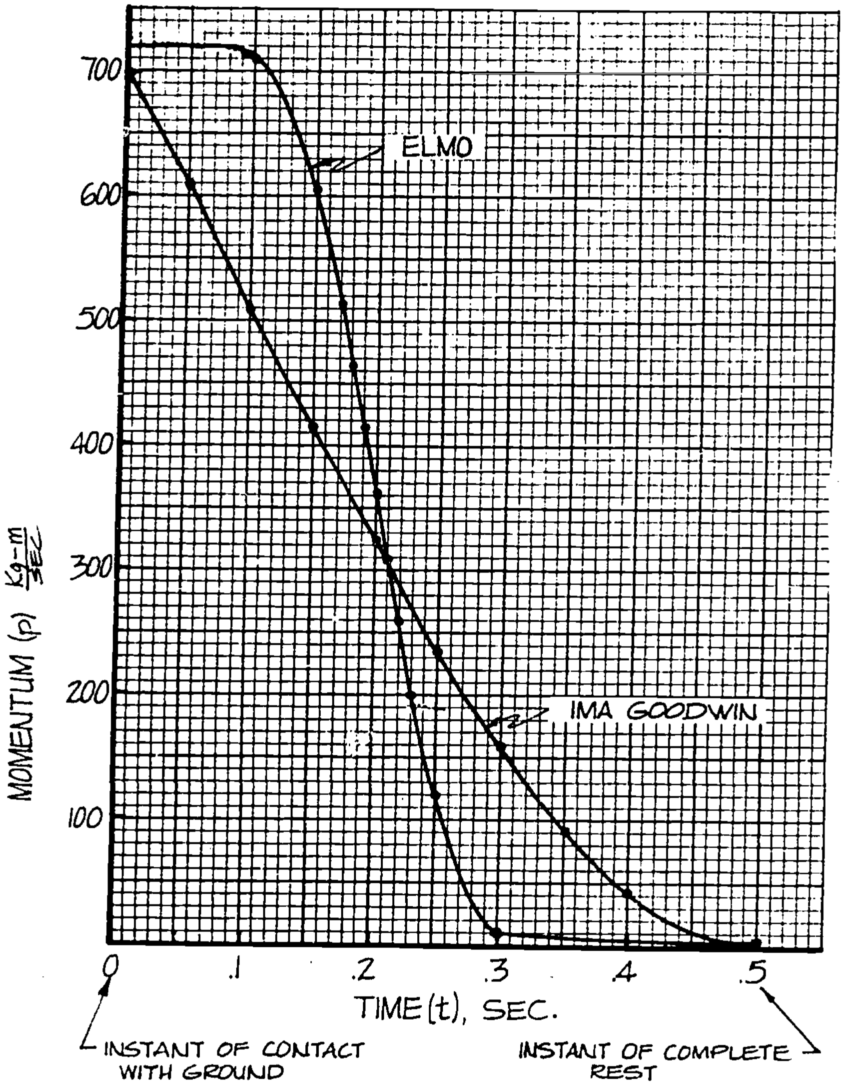
AN ILLUSTRATIVE EXAMPLE:

The graph on the following page shows time vs. momentum for two different parachute landings. The time scale of the graph starts at the instant of contact with the ground. In other words, t = 0 refers to the instant of contact with the ground, and t = .5 refers to the instant of complete rest. One parachute landing was performed by a competent person by the name of Ima Goodwin. The other one was performed by Elmo.

To demonstrate that there can be large differences between average rates and instantaneous rates, we first compute the average rates over the interval from the instant of contact to the instant of complete rest. The average rates of change of momentum for Ima and Elmo are very close to one another.

$$\text{Ima:} \quad \frac{\Delta p}{\Delta t} = \frac{700}{.5} = 1400 \ \frac{\text{kg-m}}{\text{sec}^2}$$

$$\text{Elmo:} \quad \frac{\Delta p}{\Delta t} = \frac{720}{.5} = 1440 \ \frac{\text{kg-m}}{\text{sec}^2}$$



However, Elmo suffered a broken arm, while Ima was not hurt. An examination of the graph will explain why Elmo suffered and Ms. Goodwin escaped unharmed. A portion of Elmo's graph looks steeper than Ms. Goodwin's. As we discussed previously, a steeper slope indicates a higher absolute rate of change of momentum, or in other words a greater force. We can see this merely by looking at the graph, but we cannot examine the instantaneous rates by looking at the graph. We will have to use skills associated with taking derivatives.

Recall that we must first have an equation before we can differentiate. The equations needed are given below.

Elmo's equation for t between .1 and .3 sec:

$$p = 175{,}000t^3 - 105{,}000t^2 + 15{,}750t + 10$$

Ms. Goodwin's equation for t between 0 and .5 sec:

$$p = 4{,}000t^3 - 1{,}200t^2 - 1{,}800t + 700$$

Since Elmo broke his arm, his situation would seem to be more interesting than Ima's. However, the numbers in Ima's equation will be easier to handle. Therefore, we will consider Ima's fate first and Elmo's fate later.

Now that we have selected an equation, the next step is to differentiate it.

$$p = 4{,}000t^3 - 1{,}200t^2 - 1{,}800t + 700$$
$$\frac{dp}{dt} = 3(4{,}000)t^2 - 2(1{,}200)t - 1{,}800$$
$$= 12{,}000t^2 - 2{,}400t - 1{,}800$$

This equation will tell us the instantaneous rate for any t. It is easy to calculate the rate for t = 0. All terms containing t disappear and we are left with

$$\frac{dp}{dt} = -1{,}800 \quad \text{at } t = 0.$$

Remember that Ima's average rate was -1400. The implication is that Ima experienced greater force at the moment of contact with the ground than the average force she experienced over the entire interval from 0 sec to .5 sec.

In order to illustrate a more general calculation, we will show how to calculate $\frac{dp}{dt}$ for t = .3.

<u>EXAMPLE</u>:

Calculate $\frac{dp}{dt}$ for t = .3.

<u>SOLUTION</u>:

$$\frac{dp}{dt} = 12{,}000t^2 - 2{,}400t - 1{,}800$$
$$= 12{,}000(.3)^2 - 2{,}400(.3) - 1{,}800$$
$$= 12{,}000(.09) - 720 - 1{,}800$$
$$= 1{,}080 - 2{,}520$$
$$= -1{,}440$$

Now we take up the case of Elmo. His momentum (p) for any t between .1 sec and .3 sec is given by the equation

$$p = 175{,}000t^3 - 105{,}000t^2 + 15{,}750t + 10$$

The force at any time t will be given by $\frac{dp}{dt}$.

$$\frac{dp}{dt} = (3)175{,}000t^2 - (2)105{,}000t + 15{,}750$$
$$= 525{,}000t^2 - 210{,}000t + 15{,}750$$

Now look back at the graph of p as a function of t. Notice that the graph looks very steep near t = .2 sec. By substituting t = .2 sec into the equation for $\frac{dp}{dt}$ we can determine the instantaneous force at this instant.

$$\frac{dp}{dt} = 525{,}000(.2)^2 - 210{,}000(.2) + 15{,}750$$

$$= 525{,}000(.04) - 210{,}000(.2) + 15{,}750$$

$$= 21{,}000 - 42{,}000 + 15{,}750$$

$$= -21{,}000 + 15{,}750$$

$$= -5250$$

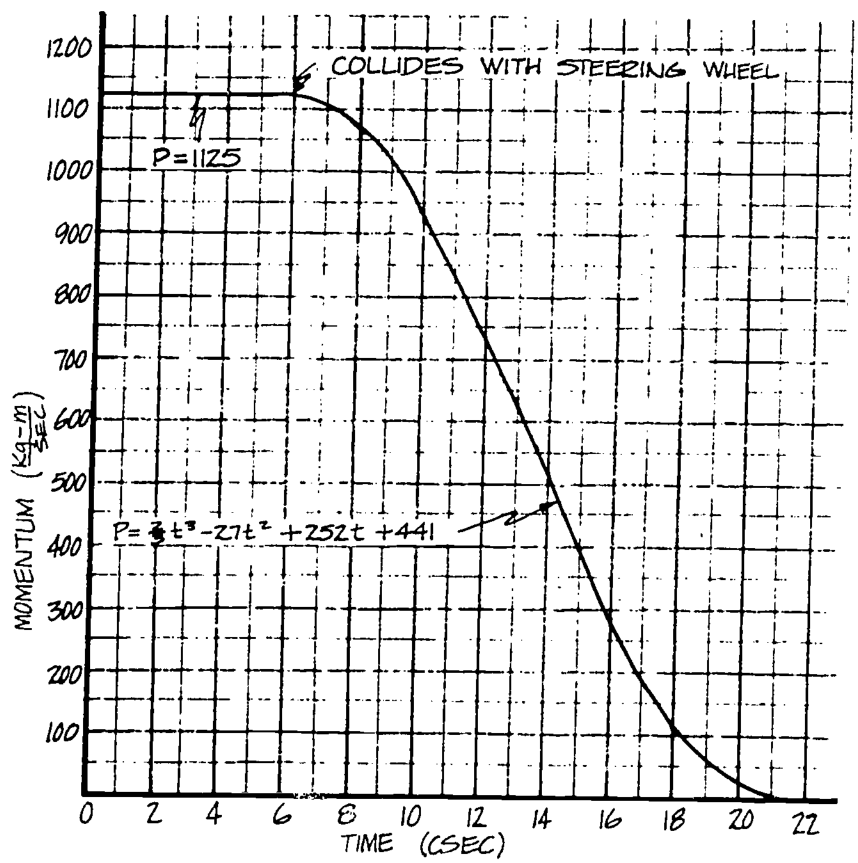
Again we compare this figure to the average rate of -1440, and we can see that Elmo experienced over $3\frac{1}{2}$ times the force at this particular instant than the average force he experienced over the entire interval.

Now its time to sum up what we've done here. By looking at the graph we can see that Elmo dissipated most of his momentum in a very short time interval. On the other hand, Ms. Goodwin spread her loss of momentum more evenly over the time interval. Elmo's pattern resulted in a tendency to experience higher instantaneous forces than Ms. Goodwin.

At this point one question is begging to be answered: What were the maximum forces experienced by the two jumpers? This topic will be taken up in the next section.

PROBLEM SET 13:

1. When Elmira Smith hit her tree, the total time it took for the car to come to a stop was 21 csec. However, Elmira stopped in less time than that. Because her seat belt was only loosely fastened, and therefore useless, she didn't start stopping when her car did. She started stopping when her chest came in contact with the steering wheel. An inspection of the graph opposite will give you the general idea. The purpose of seat belts, shoulder harnesses and air bags is to start slowing you down when your car starts slowing down. This increases Δt which decreases the ratio $\frac{\Delta p}{\Delta t}$. Since average force $= \left|\frac{\Delta p}{\Delta t}\right|$, this in turn has the effect of decreasing the average forces you might experience in an impact.



The vertical axis is scaled in the momentum units (kg-m) per (sec). The horizontal axis is scaled in units of csec.

a. How much stopping time did Elmira lose because she didn't have her seat belt tightly fastened? Remember that without units your answer is meaningless.

b. Differentiate the momentum equation to get an equation for the instantaneous force at time t. ($P = \frac{2}{3}t^3 - 27t^2 + 252t + 441$)

c. Calculate the absolute instantaneous force at time t = 10 csec (in units of $\frac{\text{kg-m}}{\text{sec-csec}}$).

2. Felix Bugdriver's impact was very similar to Elmira's. However, it only took him 2 csec to hit the steering wheel after his car hit his tree. There are two reasons why Felix's time was shorter than Elmira's. First, he was in a smaller car; consequently, he was closer to the steering wheel to start with. Second, the impact didn't last as long, which made everything happen proportionately faster. The two equations below describe Felix's momentum during the impact.

Domain (t in csec)	
$0 \le t \le 2$	$p = 1440$
$2 \le t \le 14$	$p = t^3 - 20t^2 - 28t + 1568$

a. Differentiate p = 1440.

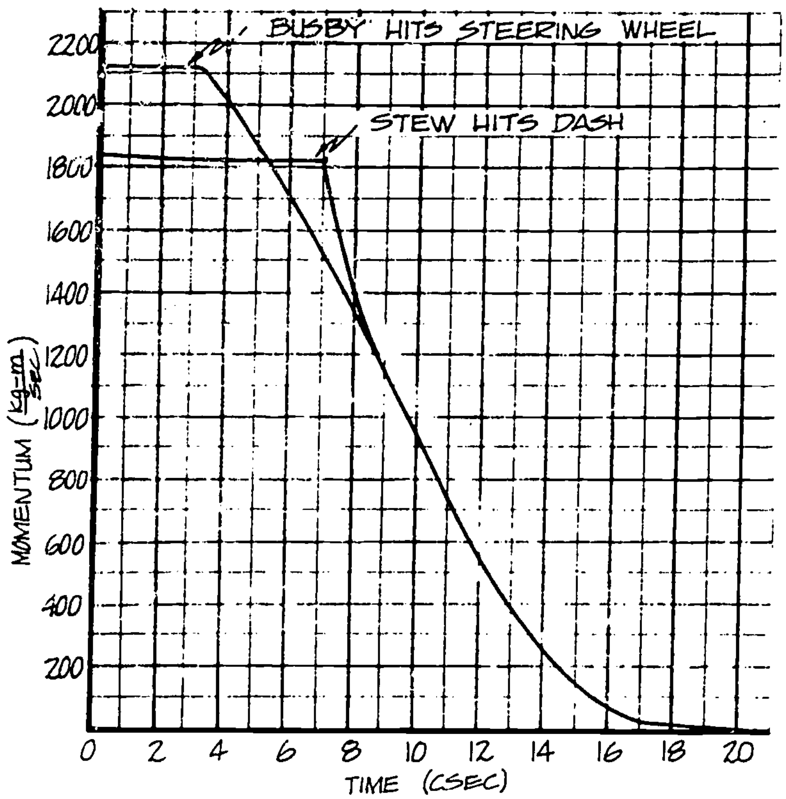
b. What was the instantaneous force on Felix at t = 1 csec?

c. Differentiate $p = t^3 - 20t^2 - 28t + 1568$ with respect to t.

d. Calculate the absolute instantaneous force on Felix at t = 5 csec.

3. Now we consider the case of Busby Berzerkly and Stewart Pid. Recall that Mr. Pid was a passenger in the right front seat. Some have called this particular position the "suicide seat." This title is easily justifiable. From a statistical point of view, it has been found in studies of thousands of accidents that the passenger in the right front seat has a much higher risk of serious injury or death than the driver. To try to understand why this is so, we will look at the situation in terms of the rate of change of momentum. Right front passengers generally collide with the dashboard. Since the dashboard is a little farther away from the seat than the steering wheel, it takes a little longer to get to it; therefore, the effective stopping time is less for the right front passenger than for the driver. The graph on the following page shows this situation.

Notice that even though Busby had more momentum to start with, the slope of Stew's graph near t = 7 csec is much steeper than Busby's is anywhere. The graphs are described by the following sets of equations.



	Domain (t in csec)	
Busby:	$0 \le t \le 3$	$p = 2135$
	$3 \le t \le 17$	$p = t^3 - 27t^2 + 41t + 2228$
	$17 \le t \le 20.5$	$p = -10t + 205$
Stewart:	$0 \le t \le 7$	$p - 1830$
	$7 \le t \le 9$	$p = -364t + 4378$
	$9 \le t \le 17$	$p = t^3 - 27t^2 + 45t + 2155$
	$17 \le t \le 20.5$	$p = \frac{-60}{7}t + 175\frac{5}{7}$

a. How much more effective stopping time did Busby have than Stew?

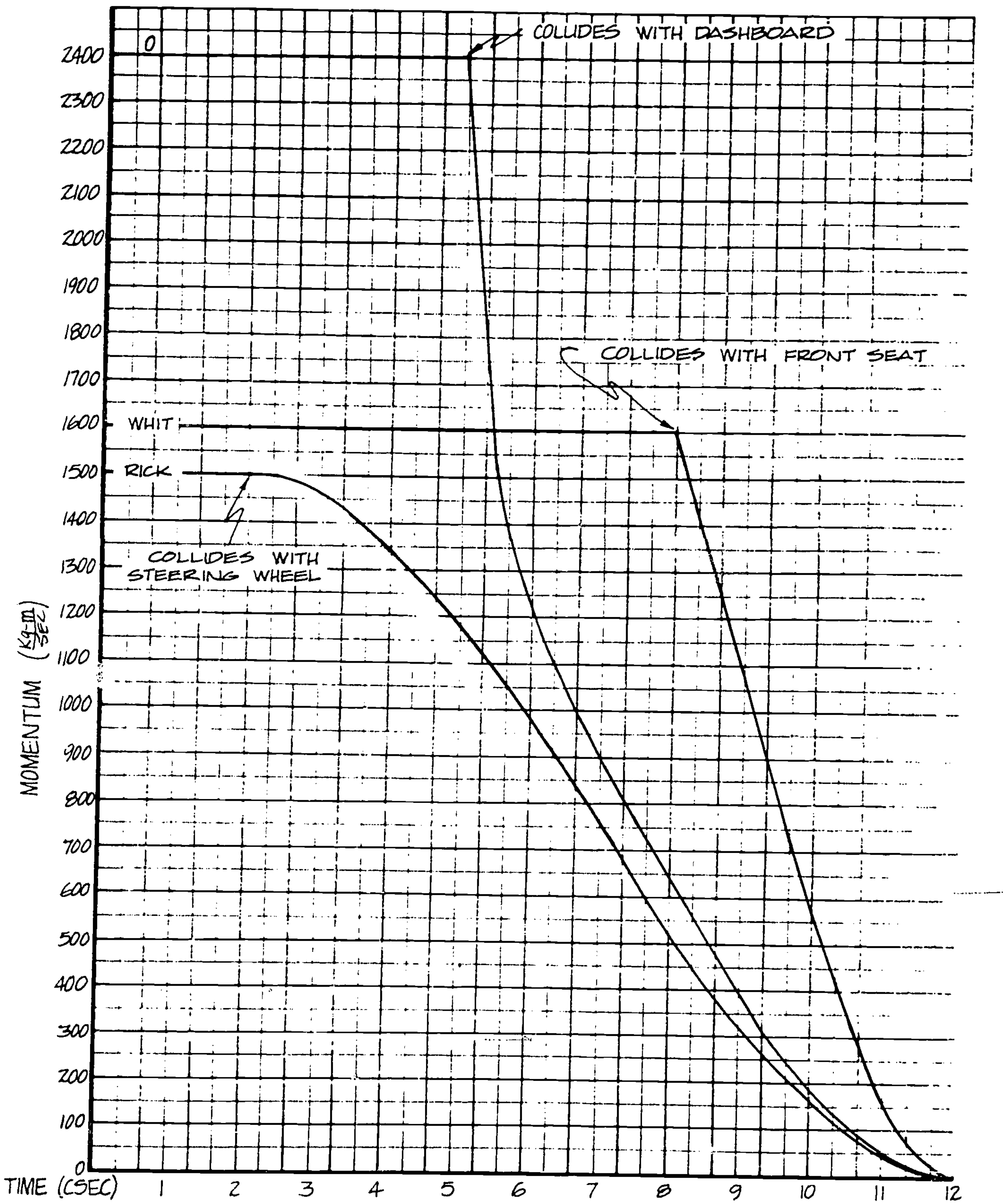
b. Find $\frac{dp}{dt}$ for each equation.

c. Calculate Busby's absolute instantaneous force at t = 10 csec (in units of $\frac{\text{kg-m}}{\text{sec-csec}}$).

d. Calculate Stewart's instantaneous force at t = 8 csec (in units of $\frac{\text{kg-m}}{\text{sec-csec}}$).

e. Look at the graph. Determine the interval where Busby's graph is steepest.

4. Now we take up the Less brothers, Rick and Whit. Remember that they had an unfortunate run-in with a rather solid bridge abutment. On the following page there is a graph of momentum vs. time for each passenger. Recall that Rick walked away while Whit and Ophelia Beese were killed rather quickly. The new factor in this



0
COLLIDES WITH DASHBOARD
COLLIDES WITH FRONT SEAT
WHIT
RICK
COLLIDES WITH STEERING WHEEL
MOMENTUM (Kg-m/SEC)
2400
2300
2200
2100
2000
1900
1800
1700
1600
1500
1400
1300
1200
1100
1000
900
800
700
600
500
400
300
200
100
0
TIME (CSEC)
1
2
3
4
5
6
7
8
9
10
11
12

75

accident is the fate of the right rear passenger. You might think that he would have a good chance to survive because he hit the back of the front seat, a much more flexible object than a dashboard. However, the car was a large size American car. Very spacious. This may be an advantage sometimes, but not in an accident. Rick had roughly twice the distance to travel before he hit a restraint. Consequently his stopping time was drastically reduced. The following equations describe the relation between time and momentum for each passenger.

	Domain (t in csec)	
Rick:	$0 \le t \le 2$	$p = 1500$
	$2 \le t \le 12$	$p = 3t^3 - 63t^2 + 216t + 1296$
O. Beese:	$0 \le t \le 5$	$p = 2400$
	$5 \le t \le 6$	$p = 450t^3 - 6750t^2 + 32{,}100t - 45{,}600$
	$6 \le t \le 12$	$p = \frac{25}{9}t^3 - 50t^2 + 2400$
Whit:	$0 \le t \le 8$	$p = 1600$
	$8 \le t \le 12$	$p = 20t^3 - 540t^2 + 4320t = 8640$

a. Calculate the instantaneous force on Rick at t = 5 csec.

b. Calculate the instantaneous force on Ophelia at t = 5 csec, using the equation for $5 \le t \le 6$.

c. Calculate the instantaneous force on Whit at t = 10 csec.

5. Finally, we consider the case of the brothers Grease. The following equations relate time to momentum for each passenger.

	Domain (t in csec)	
Axel: (driver)	$0 \le t \le 1.5$	$p = 1080$
	$1.5 \le t \le 2.0$	$p = 6t^3 - 69t^2 + 22.5t + 1181.25$
	$2.0 \le t \le 2.5$	$p = 998.25$
	$2.5 \le t \le 8$	$p = 6t^3 - 68t^2 + 96t + 1152$
Elbo: (right front)	$0 \le t \le 3$	$p = 5t^3 - 5t^2 - 105t + 225$
	$3 \le t \le 8$	$p = 4t^3 - 40t^2 - 128t + 1536$
Easy: (center rear)	$0 \le t \le 1$	$p = 960$
	$1 \le t \le 9$	$p = 3t^3 - 42t^2 + 27t + 972$

a. Calculate the instantaneous force experienced by Axel at t = 4 csec.

b. Calculate the instantaneous force experienced by Elbo at t = 4 csec.

c. Calculate the instantaneous force experienced by Easy at t = 4 csec.

6. You should recognize that the result of the accident involving the Grease brothers is in some ways a freak. For example, the driver, Axel, was killed instantly while the passenger in the suicide seat, Elbo, survived. How could this happen? Here are a few more details that we will give to you in the nature of clues.

- A broken beer bottle was smashed between Axel and the hub of the steering column.
- The only contact point that was detectable for Axel was the hub of the steering column.
- The death certificate indicated that Axel died of massive head injuries.
- Elbo had a broken hand.
- The automobile was going slow enough before the impact that the passengers had time to react.

Speculate on how the forces of impact and the pressures (force divided by area of contact) that the forces caused could be used to explain what happened.

SECTION 14: MAXIMUM FORCE

14-1 Maximum Force

In this section we are going to see how the differentiation procedure may be applied twice to squeeze a little more information out of the relationship between momentum and time. In other words, we will be differentiating the resulting equations. Before we jump into this though, we'll go through a brief review of what has happened thus far.

We started out this physics-related sequence by taking up the idea of momentum and rates of change of momentum. Next we made the connection between the instantaneous rate of change of momentum and instantaneous force. The equation below states the equality of these two physical quantities.

$$\frac{dp}{dt} = F$$

With this in mind, we examined the patterns of instantaneous force that occurred during impacts. Finally, we raised the question of how we might determine the maximum force experienced during an impact. This is an important problem. For example, suppose two impacts were similar in all features except one. The one difference was that in one impact there was a higher maximum instantaneous force. It seems reasonable that the impact with the higher maximum instantaneous force would be more likely to produce an injury.

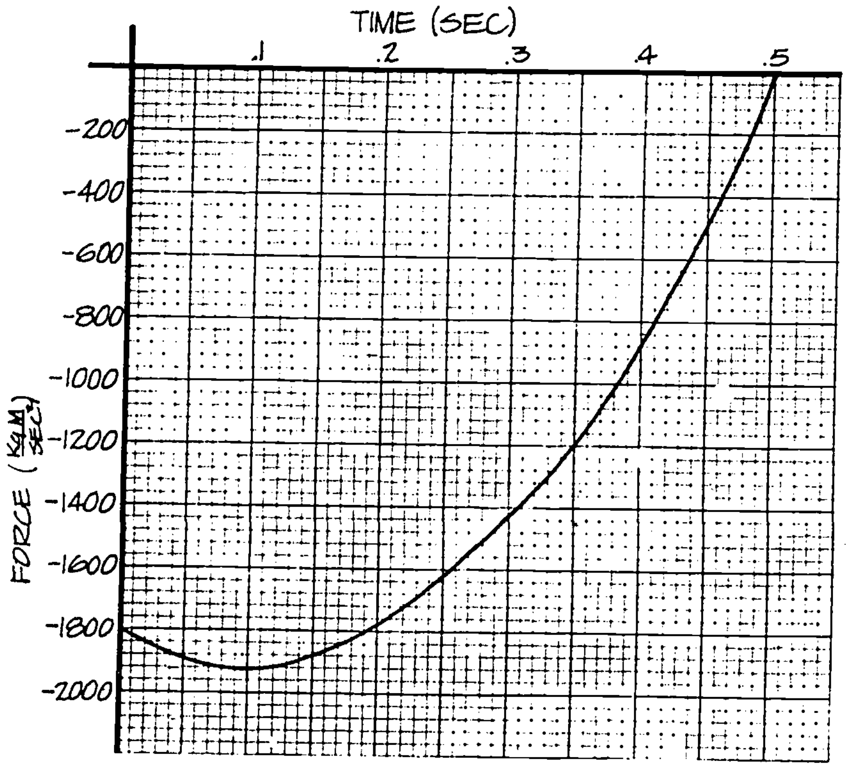
How shall we go about determining the maximum instantaneous force? We shall apply the maxima-minima technique to the force equation. In the previous section we examined the change in momentum which occurred in two parachute landings. Here we will demonstrate how to find the maximum force experienced by Ms. Goodwin. Her momentum was related to time by the equation

$$p = 4{,}000t^3 - 1{,}200t^2 - 1800t + 700$$

The derivative of p with respect to t is the force at time t.

$$\frac{dp}{dt} = 12{,}000t^2 - 2{,}400t - 1{,}800$$
$$F = 12{,}000t^2 - 2{,}4000t - 1{,}800$$

You should recognize that this last equation is a simple quadratic. The following is a graph of this equation over the interval $0 \leq t \leq .5$ seconds.



This is a graph of the instantaneous force experienced by Ima during her impact with the ground. We can easily see that there is a minimum associated with $t = .1$ sec. An inspection of the graph will reveal that the force at this instant was -1920. Since we won't always want to take the time to graph the force equation we will outline the analytic procedure. We will treat the force equation just as we would any other equation and apply the max-min procedure to it.

First we take the derivative.

$$F = 12{,}000t^2 - 2{,}400t - 1{,}800$$
$$\frac{dF}{dt} = 2(12{,}000)t = 2{,}400$$
$$= 24{,}000t - 2{,}400$$

The second step is to set the derivative equal to zero and solve for t.

$$0 = 24{,}000t - 2{,}400$$
$$2400 = 24{,}000t$$
$$\frac{2400}{24{,}000} = t$$
$$.1 = t$$

We see that Ms. Goodwin experienced the maximum force at the instant t = .1 sec. All that remains is to substitute t = .1 back into the force equation. This will tell us what the force was at the instant t = .1 sec.

$$\begin{aligned} F &= 12{,}000t^2 - 2{,}400t - 1{,}800 \\ &= 12{,}000(.1)^2 - 2{,}400(.1) - 1{,}800 \\ &= 12{,}000(.01) - 2{,}400(.1) - 1{,}800 \\ &= 120 - 240 - 1{,}800 \\ &= -1920\ \frac{\text{kg-m}}{\text{sec}^2} \end{aligned}$$

Note that this value is exactly the same as the graphical solution and the algebraic approach saved us much time and effort. You can partially check this by referring back to Section 13. You will see that the absolute value of this force (i.e., 1920 kg-m/sec^2) exceeds the two absolute forces calculated in that section.

14-2 A Summarized Maximum Force Procedure

Since there are several steps in the procedure of determining the maximum absolute force during an impact, we have developed a step-by-step procedure.

1. Differentiate the momentum equation with respect to t.

   EXAMPLE: $p = 3t^3 - 2t^2 + t + 4$

   $$\frac{dp}{dt} = 9t^2 - 4t + 1$$

2. Call this equation the force equation.

   EXAMPLE: $F = 9t^2 - 4t + 1$

3. Differentiate the force equation with respect to time.

   EXAMPLE: $\frac{dF}{dt} = 18t - 4$

4. Equate to zero and solve for t.

   EXAMPLE: $$\begin{aligned} 0 &= 18t - 4 \\ 4 &= 18t \\ \frac{4}{18} &= t \\ \frac{2}{9} &= t \end{aligned}$$

5. Substitute this value of t back into the force equation. This gives the force at the instant of maximum absolute force.

   EXAMPLE: $$\begin{aligned} F &= 9t^2 - 4t + 1 \\ &= 9\left(\frac{2}{9}\right)^2 - 4\left(\frac{2}{9}\right) + 1 \\ &= 9\left(\frac{4}{9^2}\right) - \frac{8}{9} + \frac{9}{9} \\ &= \frac{4}{9} - \frac{8}{9} + \frac{9}{9} \\ &= \frac{5}{9} \end{aligned}$$

PROBLEM SET 14:

1. Elmira's momentum as a function of time is described by the equation

$$p = \frac{2}{3}t^3 - 27t^2 + 252t + 441$$

for any t such that 6 csec $\leq$ t $\leq$ 21 csec.

a. Show steps which confirm that the instantaneous force at any time t within the given domain is given by the equation

$$F = 2t^2 - 54t + 252$$

b. Find $\frac{dF}{dt}$.

c. Equate $\frac{dF}{dt}$ = - and solve for t. This tells you the instant that the force was a minimum (or maximum absolute value).

d. Substitute your answer to Part c into the force equation of Part a.

e. What is this force (answer to d) called?

2. Follow the procedure exactly as outlined in Problem 1 to find the maximum absolute force experienced by Felix Bugdriver in the interval 2 csec $\leq$ t $\leq$ 14 csec.

$$p = t^3 - 20t^2 - 28t + 1568$$

3. The equations which describe Busby's accident are given below.

Busby: Domain (t in csec)

$0 \leq t \leq 3$	$p = 2135$
$3 \leq t \leq 17$	$p = t^3 - 27t^2 + 41t + 2228$
$17 \leq t \leq 20.5$	$p = -10t + 205$

Stewart:

$0 \leq t \leq 7$	$p = 1830$
$7 \leq t \leq 9$	$p = -364t + 4378$
$9 \leq t \leq 17$	$p = t^3 - 27t^2 + 45t + 2155$
$17 \leq t \leq 20.5$	$p = \frac{-60}{7}t + 175\,\frac{5}{7}$

Determine the maximum force experienced by each passenger.

4. The equations for Rick Less, O. Beese and Whit Less are listed below.

	Domain (t in csec)	
Rick:	$0 \leq t \leq 2$	$p = 1500$
	$2 \leq t \leq 12$	$p = 3t^3 - 63t^2 + 216t + 1296$
Ophelia:	$0 \leq t \leq 5$	$p = 2400$
	$5 \leq t \leq 6$	$p = 450t^3 - 6750t^2 + 32{,}100t - 45600$
	$6 \leq t \leq 12$	$p = \frac{25}{9}t^3 - 50t^2 + 2400$
Whit:	$0 \leq t \leq 8$	$p = 1600$
	$8 \leq t \leq 12$	$p = 20t^3 - 540t^2 + 4320t - 8640$

a. Determine the maximum force experienced by Rick Less.

b. Determine the <u>time</u> when Ophelia experienced the maximum force. To find the force at that time, you may either substitute the appropriate t into the force equation or estimate the slope of the momentum curve from the graph in Problem Set 15.

c. Determine the maximum force experienced by Whit Less.

5. The equations for the brothers Grease are listed below.

Axel:	$0 \le t \le 1.5$	$p = 1080$
(driver)	$1.5 \le t \le 2.0$	$p = 6t^3 - 69t^2 + 22.5t + 1181.25$
	$2.0 \le t \le 2.5$	$p = 998.25$
	$2.5 \le t \le 8$	$p = 6t^3 - 78t^2 + 96t + 1152$
Elbo:	$0 \le t \le 3$	$p = 5t^3 - 5t^2 - 105t + 225$
(right front)	$3 \le t \le 8$	$p = 4t^3 - 40t^2 - 128t + 1436$
Easy:	$0 \le t \le 1$	$p = 960$
(center rear)	$1 \le t \le 9$	$p = 3t^3 - 42t^2 + 2$[illegible] 972

a. Determine the time t when the force on Axel was the greatest. Then calculate the force for the nearest whole <u>csec</u>.

b. Calculate the maximum force on Elbo. The instants of maximum force will both be fractions. You may use the nearest whole number values of t in your calculations.

c. Determine the greatest force exerted on Easy. Again, you may use the nearest integral value of t in the calculations.

6. a. Make an ordered list of the people involved and their maximum forces. Start with the lowest and go to the highest. For example,

112.5 Elmira Smith

$161\frac{1}{3}$ Felix

etc.

b. Divide the list into three parts. Label the parts
"Probably Survive"
"Possibly Die--Serious Injury"
"Probably Die"

REVIEW PROBLEM SET 15:

1. Differentiate the following functions with respect to t.

a. $y = t - \frac{t^3}{6} + \frac{t^5}{120}$

b. $y = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720}$

c. $y = t^2 + 22t^4 - \frac{t^{16}}{32}$

d. $y = .3t^{100} - \frac{7}{8}t^{80} + \frac{40}{49}t^7$

2. Find the values of t which give maximum or minimum y-values for the following functions. Substitute these t-values back into the functions to find the maximum or minimum y-values.

a. $y = \frac{1}{3}t^2 + \frac{1}{2}t^2 - 2t + 1$

b. $y = t^3 - 3t - 1$

c. $y = t^4 - \frac{4}{3}t^3 - 4t^2 + 3$

d. $y = \frac{3}{2}t^2 - 3t - \frac{3}{2}$

e. $y = \frac{1}{2}t^2 + 4t + 10$

3. Norbert's niece Hermionie has read about Galileo's famous experiment of dropping various objects off the Leaning Tower of Pisa. Hermionie decided to repeat this experiment, so she took various objects which were lying about the house, brought them up to the attic and chucked them out the window.

a. Her father's granite paper weight had a mass of 2 kg. When it struck the ground, it was traveling at 5 m/sec. What was its momentum?

b. When the paper weight struck the petunia bed, it buried itself halfway into the ground, coming to a complete stop in 1 dsec (.1 sec). What was the absolute value of the average force on the paper weight during its collision with the ground?

c. The momentum equation for the paper weight's impact with the petunia bed is

$$p = 10t^3 - 12t^2 - 6t + 8, \text{ t in dsec}$$

where 10 dsec = 1 sec. What is the force equation for the paper weight?

d. At what time, t, is the absolute instantaneous force at a maximum?

e. What is the maximum absolute force? Express in kg-m/sec^2.

4. The Ming vase from the hall table had a mass of 1.5 kg. When it struck the ground, it was also traveling at 4 m/sec. What was the momentum?

5. Hermionie made a parachute out of part of an old bedspread and attached it to the goldfish bowl before launching the fish. The goldfish bowl (fish, water and all) had a mass of 8 kg. Because of the parachute, it was only traveling at 3 m/sec when it hit the ground.

a. What was its momentum?

b. The goldfish bowl, wonder of wonders, didn't break. It buried itself in the petunia bed next to the paper weight, coming to a stop in 1 dsec (.1 sec). What was the absolute value of the average force during the impact? Express in $\frac{\text{kg-m}}{\text{sec}^2}$.

c. The following momentum equation describes the goldfish bowl's arrival among the petunias. t is in dsec (10 dsec = 1 sec).

$$p = 48t^3 - 72t^2 + 24$$

What is the force equation for the goldfish bowl?

d. At what time is the absolute instantaneous force at a maximum?

e. What is the maximum absolute force? Express in $\frac{\text{kg-m}}{\text{sec}^2}$.